Introduction to Regularity Theory

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References

- M. Giaquinta, Multiple integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Annals of Mathematics Studies 105, Princeton University Press (1983)
- M. Giaquinta, Introduction to Regularity Theory for Nonlinear Elliptic Systems, Lectures in Mathematics, ETH Zurich (1993)

Let us consider

$$\mathrm{div}(\mathcal{A}(x)Du) = \mathcal{F} + \mathrm{div}\mathcal{G} \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1}$$

- Ω bounded open set in $\mathbb{R}^n$
- $u : \Omega \to \mathbb{R}$
- $\mathcal{G} : \Omega \to \mathbb{R}^n$
- $\mathcal{F} : \Omega \to \mathbb{R}$
- $\mathcal{A} : \Omega \to \mathbb{R}^{n\times n}$ measurable matrix valued function

Example

$$\Delta u = f$$

The Assumptions

There exist constants $\alpha, \beta > 0$ such that

$$\alpha|\xi|^2 \leqslant \langle \mathcal{A}(x)\xi, \xi\rangle \tag{2}$$

$$|\mathcal{A}(x)| \leqslant \beta \tag{3}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$.

$$\mathcal{F} \in L^2(\Omega) \tag{4}$$

$$\mathcal{G} \in L^2(\Omega, \mathbb{R}^n) \tag{5}$$

Let $p \in \mathbb{R}$, $1 \leqslant p < \infty$,

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R}, f \text{ measurable and } |f|^p \text{ is integrable}\}$$

Classical solutions **versus** distributional solutions

- u is a classical solution if $u \in C^2(\Omega)$ satisfies (1) pointwise
- u is a distributional (weak) solution if $u \in W^{1,2}(\Omega)$ is such that

$$\int_\Omega \langle \mathcal{A}(x)Du, D\varphi\rangle \, dx = \int_\Omega \mathcal{F}\varphi \, dx + \int_\Omega \langle \mathcal{G}, D\varphi\rangle \, dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

A classical solution is a weak solution (integration by parts)

We say that $u \in L^p(\Omega)$ has weak derivatives $(v_1, ..., v_n) = Du$ in L^p if for all $i = 1, ..., n$

$$\int_\Omega u D_i \varphi \, dx = - \int_\Omega v_i \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

The class of functions $u \in L^p(\Omega)$ that possess weak derivatives in L^p is denoted by $W^{1,p}(\Omega)$.

The Sobolev space, denoted by $W^{k,p}(\Omega)$ $(1 \leqslant p \leqslant \infty)$ is the linear space consisting of all functions having weak derivatives: $D^\alpha f \in L^p(\Omega)$ for each α, $|\alpha| \leqslant k$. $W^{k,p}(\Omega)$ is equipped with the norm:

$$\|f\|_{k,p} = \left(\sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha f(x)|^p \, dx \right)^{\frac{1}{p}} \quad \text{if } p \neq \infty$$

and

$$\|f\|_{k,\infty} = \max_{|\alpha| \leqslant k} \|D^\alpha f\|_\infty$$

Standard Program

1. Existence of a weak solution
2. Regularity of the weak solution
3. If the weak solution is sufficiently regular then is a classical solution.

A first step: the Caccioppoli inequality

Theorem (Caccioppoli inequality)

Let $u \in W^{1,2}(\Omega)$ be a weak solution to the equation (1) *under the assumptions* (2), (3), (4), (5). *Then there exists a constant $c = c(\alpha, \beta, n)$ such that the following inequality*

$$\begin{aligned}\int_{B_\rho} |Du|^2 \;\leqslant\; & \frac{c}{(r-\rho)^2} \int_{B_r} |u - u_r|^2 + c(r-\rho)^2 \int_{B_r} |\mathcal{F}|^2 \\ + \;& c \int_{B_r} |\mathcal{G}|^2\end{aligned}$$

holds for every balls $B_\rho \subset B_r \subset \Omega$ ($u_r = \frac{1}{|B_r|} \int_{B_r} u dx$).

Sketch of the proof

Let $B_\rho \subset B_r \subset \Omega$ and let $\eta \in C_0^\infty(B_r)$ be a cut-off function s.t.

$$0 \leqslant \eta \leqslant 1, \quad \eta = 1 \text{ in } B_\rho, \quad |D\eta| \leqslant \frac{c}{r-\rho}.$$

Using $\varphi = \eta^2(u - u_r)$ as test function in (1), we get

$$\int_{B_r} \eta^2 \langle \mathcal{A}(x) Du, Du \rangle + 2 \int_{B_r} \eta \langle \mathcal{A}(x) Du, D\eta (u - u_r) \rangle$$

$$= \int_{B_r} \eta^2 \mathcal{F}(u - u_r) + \int_{B_r} \langle \eta^2 \mathcal{G}, Du \rangle + 2 \int_{B_r} \eta \langle \mathcal{G}, D\eta (u - u_r) \rangle$$

$$\int_{B_r} \eta^2 \langle \mathcal{A}(x) Du, Du \rangle \leqslant 2 \int_{B_r} \eta |\mathcal{A}(x)||Du||D\eta||u - u_r|$$
$$+ \int_{B_r} \eta^2 |\mathcal{F}||u - u_r| + \int_{B_r} \eta^2 |\mathcal{G}||Du| + 2 \int_{B_r} |\mathcal{G}||D\eta||u - u_r|$$

The growth assumption (3) and ellipticity assumption (2) yield

$$\alpha \int_{B_r} \eta^2 |Du|^2 \leqslant \frac{c(\beta)}{r - \rho} \int_{B_r} \eta |Du||u - u_r| + \int_{B_r} \eta^2 |\mathcal{F}||u - u_r|$$
$$+ \int_{B_r} \eta^2 |\mathcal{G}||Du| + \frac{c}{r - \rho} \int_{B_r} |\mathcal{G}||u - u_r|$$

Using Young's inequality : $ab \leqslant \varepsilon a^2 + \frac{b^2}{\varepsilon}$

$$\alpha \int_{B_r} \eta^2 |Du|^2 \leqslant \varepsilon \int_{B_r} \eta^2 |Du|^2 + \frac{c(\beta)}{\varepsilon (r-\rho)^2} \int_{B_r} |u - u_r|^2 + c(r-\rho)^2 \int_{B_r} \eta |\mathcal{F}|^2 + c \int_{B_r} |\mathcal{G}|^2$$

Choosing $\varepsilon = \frac{\alpha}{2}$ we have

$$\alpha \int_{B_r} \eta^2 |Du|^2 \leqslant \frac{c(\alpha,\beta)}{(r-\rho)^2} \int_{B_r} |u - u_r|^2 + c(r-\rho)^2 \int_{B_r} |\mathcal{F}|^2 + c \int_{B_r} |\mathcal{G}|^2$$

We conclude recalling that $\eta = 1$ on B_ρ.

Hilbert regularity

Our aim is to prove the higher differentiability of the weak solutions.
Let $u \in W^{1.2}_{loc}(\Omega)$ be a solution to

$$\Delta u = f$$

Assume that $u \in W^{2,2}_{loc}(\Omega)$ and let us differentiate the equation

$$\frac{\partial}{\partial x_i}(\Delta u) = \Delta \left(\frac{\partial u}{\partial x_i} \right) = \frac{\partial}{\partial x_i}(f)$$

so $v = \frac{\partial u}{\partial x_i}$ is a solution to

$$\mathrm{div}(Dv) = \mathrm{div} f$$

The function v satisfies Caccioppoli inequality and so:

$$\int_{B_{\frac{r}{2}}} |Dv|^2 \leqslant \frac{c}{r^2} \int_{B_r} |v|^2 + c \int_{B_r} |f|^2$$

i.e.

$$\int_{B_{\frac{r}{2}}} \left| D \left(\frac{\partial u}{\partial x_i} \right) \right|^2 \leqslant \frac{c}{r^2} \int_{B_r} \left| \frac{\partial u}{\partial x_i} \right|^2 + c \int_{B_r} |f|^2$$

So, if u possesses weak second derivatives, we have

$$\int_{B_{\frac{r}{2}}} |D^2 u|^2 \leqslant \frac{c}{r^2} \int_{B_r} |Du|^2 + c \int_{B_r} |f|^2$$

Remove the extra assumption $u \in W^{2,2}_{loc}(\Omega)$
Let us fix a compact set $\Omega' \subset \Omega$, and for a smooth kernel $\rho \in C_0^\infty(B_1(0))$, with $\rho \geq 0$ and $\int_{B_1(0)} \varphi = 1$. Let us consider the corresponding family of mollifiers $(\rho_\varepsilon)_{\varepsilon>0}$. Take a sequence of mollifier ρ_ε and observe that

1. $\Delta(u * \rho_\varepsilon) = f * \rho_\varepsilon$
2. $u * \rho_\varepsilon \in W^{2,2}$

By Caccioppoli estimate

$$\int_{B_{\frac{r}{2}}} |D^2(u * \rho_\varepsilon)|^2 \leqslant \frac{c}{r^2} \int_{B_r} |D(u * \rho_\varepsilon)|^2 + c \int_{B_r} |f * \rho_\varepsilon|^2 \quad (6)$$

Since $D(u * \rho_\varepsilon) \to Du$ strongly in L^2 and $f * \rho_\varepsilon \to f$ strongly in L^2, taking the $\limsup$ as $\varepsilon \to 0$ in estimate (6), we get

$$\limsup_{\varepsilon \to 0} \int_{B_{\frac{r}{2}}} |D^2(u * \rho_\varepsilon)|^2 \leqslant \frac{c}{r^2} \int_{B_r} |Du|^2 + c \int_{B_r} |f|^2$$

Therefore $u * \rho_\varepsilon \rightharpoonup u$ in $W^{2,2}_{loc}(\Omega)$

$$\int_{B_{\frac{r}{2}}} |D^2 u|^2 \leqslant \frac{c}{r^2} \int_{B_r} |Du|^2 + c \int_{B_r} |f|^2$$

Theorem

Let $u \in W^{1,2}(\Omega)$ *be a weak solution to the equation* (1) *under the ellipticity assumption* (2). *Assume moreover*

$$\mathcal{A} \in Lip_{loc}(\Omega) \tag{7}$$

$$\mathcal{F} \in L^2_{loc}(\Omega) \tag{8}$$

$$\mathcal{G} \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n) \tag{9}$$

Then $u \in W^{2,2}_{loc}(\Omega)$ *and the second order Caccioppoli inequality*

$$\int_{B_{\frac{r}{2}}} |D^2 u|^2 \leqslant c \left[\int_{B_r} |D\mathcal{G}|^2 + \frac{c}{r^2} \int_{B_r} |Du|^2 + c \int_{B_r} |\mathcal{F}|^2 \right]$$

holds for every ball $B_r \subset \Omega$.

Tools: Difference quotient.
Given a function $f \in L^p(\Omega)$, for $h \in \mathbb{R}^n$ and $s \in \{1, ..., n\}$ we define:

$$\Delta_{s,h} f = \frac{\tau_{s,h} f}{h} = \frac{f(x + he_s) - f(x)}{h},$$

where $e_s \in \mathbb{R}^n$ is the unit vector $(0, ..., 0, 1, 0, ...0)$ and 1 in the s-th position.
The basic properties are:

1. $\tau_{s,h}(fg)(x) = f(x + e_s h)\tau_{s,h} g(x) + g(x)\tau_{s,h}(f)(x)$
2. $\int_\Omega \tau_{s,h}(f)(x)g(x) = -\int_\Omega \tau_{s,h}(g)(x)f(x)$ (if at least one between f and g have compact support in Ω)

1. $f \in W^{1,p}(\Omega) \Rightarrow D(\tau_{s,h} f) = \tau_{s,h} Df$
2. $f \in L^p(\Omega)$ then

$$\frac{\partial f}{\partial x_s} \in L^p_{loc}(\Omega) \Longleftrightarrow \int_{\Omega'} |\tau_{s,h}(f)|^p \leqslant c(\Omega')|h|^p \quad \forall \Omega' \subset \Omega$$

Sketch of the proof. Case $\mathcal{F} = \mathcal{G} = 0$.
Let $B_r \subset \Omega$ and let $\eta \in C_0^\infty(B_r)$ be a cut-off function s.t,.

$$0 \leqslant \eta \leqslant 1, \;\; \eta = 1 \;\text{ in } B_{\frac{r}{2}}, \;\; |D\eta| \leqslant \frac{c}{r}.$$

Using $\varphi = \tau_{s,-h}(\eta^2 \tau_{s,h} u)$ as test function in (1), we get

$$\int_{B_r} \langle \mathcal{A}(x) Du, \tau_{s,-h}(D(\eta^2 \tau_{s,h} u)\rangle = 0$$

Since η has compact support, by Property 2 of different quotient

$$\int_{B_r} \langle \tau_{s,h}(\mathcal{A}(x)Du), D(\eta^2 \tau_{s,h} u) \rangle = 0$$

i.e. by Property 1

$$\int_{B_r} \langle \tau_{s,h}(\mathcal{A}(x))Du, D(\eta^2 \tau_{s,h} u) \rangle + \int_{B_r} \langle \mathcal{A}(x) \tau_{s,h}(Du), D(\eta^2 \tau_{s,h} u) \rangle = 0$$

$$
\begin{aligned}
\int_{B_r} \eta^2 \langle \mathcal{A}(x)\tau_{s,h}Du, \tau_{s,h}Du\rangle &= -2\int_{B_r} \eta\langle \mathcal{A}(x)\tau_{s,h}Du, D\eta\tau_{s,h}u\rangle \\
&- \int_{B_r} \eta^2\langle \tau_{s,h}(\mathcal{A}(x))Du, \tau_{s,h}Du\rangle \\
&- 2\int_{B_r} \eta\langle \tau_{s,h}(\mathcal{A}(x))Du, D\eta\tau_{s,h}u\rangle
\end{aligned}
$$

which implies

$$
\begin{aligned}
\int_{B_r} \eta^2 \langle \mathcal{A}(x)\tau_{s,h}Du, \tau_{s,h}Du\rangle &\leqslant 2\int_{B_r} \eta|D\eta||\mathcal{A}(x)||\tau_{s,h}Du||\tau_{s,h}u| \\
&+ \int_{B_r} \eta^2|\tau_{s,h}(\mathcal{A}(x))||Du||\tau_{s,h}Du| \\
&+ 2\int_{B_r} \eta|D\eta||\tau_{s,h}(\mathcal{A}(x))||Du||\tau_{s,h}u|
\end{aligned}
$$

Using

- The ellipticity condition (2)
- The boundedness assumption (3) (black term)
- The Lipschitz continuity of $\mathcal{A}(x)$ (blue and red)

we get

$$
\begin{aligned}
\alpha \int_{B_r} \eta^2 |\tau_{s,h} Du|^2 \leqslant \; & c|h| \int_{B_r} \eta^2 |Du||\tau_{s,h} Du| \\
& + \; c(\beta) \int_{B_r} \eta |\tau_{s,h} Du||D\eta||\tau_{s,h} u| \\
& + \; c|h| \int_{B_r} \eta |Du||D\eta||\tau_{s,h} u|
\end{aligned}
$$

Young's inequality

$$\begin{aligned}\alpha \int_{B_r} \eta^2 |\tau_{s,h} Du|^2 &\leqslant \varepsilon \int_{B_r} \eta^2 |\tau_{s,h} Du|^2 + c_\varepsilon |h|^2 \int_{B_r} \eta^2 |Du|^2 \\ &+ \varepsilon \int_{B_r} \eta^2 |\tau_{s,h} Du|^2 + c_\varepsilon \int_{B_r} |D\eta|^2 |\tau_{s,h} u|^2 \\ &+ \varepsilon \int_{B_r} \eta^2 |\tau_{s,h} u|^2 + c|h|^2 \int_{B_r} \eta^2 |Du|^2\end{aligned}$$

i.e.

$$\begin{aligned}\alpha \int_{B_r} \eta^2 |\tau_{s,h} Du|^2 &\leqslant 2\varepsilon \int_{B_r} \eta^2 |\tau_{s,h} Du|^2 \\ &+ c_\varepsilon \int_{B_r} |D\eta|^2 |\tau_{s,h} u|^2 + c|h|^2 \int_{B_r} \eta^2 |Du|^2\end{aligned}$$

Choosing $\varepsilon = \frac{\alpha}{4}$ we can reabsorb the first integral

$$\alpha \int_{B_r} \eta^2 |\tau_{s,h} Du|^2 \leqslant c \int_{B_r} |D\eta|^2 |\tau_{s,h} u|^2 + c|h|^2 \int_{B_r} \eta^2 |Du|^2$$

The assumption

- $u \in W^{1,2}(\Omega)$
- The Property 4

imply

$$\alpha \int_{B_r} \eta^2 |\tau_{s,h} Du|^2 \leqslant \frac{c|h|^2}{r^2} \int_{B_r} |Du|^2$$

and so

$$\alpha \int_{B_{\frac{r}{2}}} \frac{|\tau_{s,h} Du|^2}{|h|^2} \leqslant \frac{c}{r^2} \int_{B_r} |Du|^2$$

$$\Rightarrow \int_{B_{\frac{r}{2}}} |D^2 u|^2 \leqslant \frac{c}{r^2} \int_{B_r} |Du|^2$$

Arguing inductively we can show

Theorem

Let $u \in W^{1,2}(\Omega)$ *be a weak solution to the equation* (1) *under the ellipticity condition* (2)*. Assume moreover*

- $\mathcal{A} \in C^{k,1}_{loc}(\Omega)$ *i.e.* $D^k A \in Lip_{loc}(\Omega)$
- $\mathcal{F} \in W^{k,2}_{loc}(\Omega)$
- $\mathcal{G} \in W^{k+1,2}_{loc}(\Omega, \mathbb{R}^n)$

Then $u \in W^{k+2,2}_{loc}(\Omega)$.

Schauder estimates

Analysis of the Hölder regularity of weak solutions assuming some Hölder regularity for the data (coefficients and right hand side)

Morrey spaces

- Ω bounded regular domain, i.e. there exists a positive constant A such that

$$|\Omega \cap B_\rho(x)| > A\rho^n,$$

$\forall x \in \overline{\Omega}$, $\forall \rho < \mathrm{diam}\Omega$.

- Let $p \geqslant 1$, $\lambda > 0$

The Morrey space $L^{p,\lambda}$ is defined as

$$L^{p,\lambda}(\Omega) = \left\{ u \in L^p(\Omega) : \sup_{x_0 \in \Omega, 0<\rho<\mathrm{diam}\Omega} \frac{1}{\rho^\lambda} \int_{B_\rho(x_0)\cap\Omega} |u|^p \, dx < +\infty \right\} \tag{10}$$

it is a Banach space with the norm

$$\|u\|_{L^{p,\lambda}(\Omega)} = \sup_{x_0 \in \Omega, 0<\rho<\mathrm{diam}\Omega} \frac{1}{\rho^\lambda} \int_{B_\rho(x_0)\cap\Omega} |u|^p \, dx$$

Remarks

- $u \in L^{p,\lambda}(\Omega) \iff \int_{B_{\rho(x_0)\cap\Omega}} |u|^p dx \leqslant c\rho^\lambda$
- $L^\infty(\Omega) \subset L^{p,\lambda}(\Omega) \subset L^p(\Omega)$
- $L^{p,0}(\Omega) \simeq L^p(\Omega)$
- $L^{p,n}(\Omega) \simeq L^\infty(\Omega)$
- $L^{p,\lambda}(\Omega) = 0$, if $\lambda > n$.

Campanato spaces

Let Ω be a bounded regular domain, $p \geqslant 1$ and $\lambda > 0$.
The Campanato space $\mathcal{L}^{p,\lambda}$ is defined as:

$$\mathcal{L}^{p,\lambda}(\Omega) = \tag{11}$$

$$= \left\{ u \in L^p(\Omega) : \sup_{x_0 \in \Omega, 0 < \rho < \operatorname{diam}\Omega} \frac{1}{\rho^\lambda} \int_{B_\rho(x_0)\cap\Omega} |u - u_{x_o,\rho}|^p \, dx < +\infty \right\} \tag{12}$$

the quantity

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \sup_{x_0 \in \Omega, 0 < \rho < \operatorname{diam}\Omega} \frac{1}{\rho^\lambda} \int_{B_\rho(x_0)\cap\Omega} |u - u_{x_0,\rho}|^p \, dx$$

is a seminorm (constant functions have seminorm 0).
The norm is defined as

$$|||u|||_{\mathcal{L}^{p,\lambda}(\Omega)} = \|u\|_{L^p} + [u]_{\mathcal{L}^{p,\lambda}(\Omega)}$$

Remarks

- $\mathcal{L}^{p,\lambda}(\Omega) \simeq L^{p,\lambda}(\Omega)$, $0 < \lambda < n$
- $L^\infty(\Omega) \subset \mathcal{L}^{p,n}(\Omega)$
  Example $\log|x| \in \mathcal{L}^{1,1}((0,1]) \setminus L^\infty((0,1])$
- Campanato Theorem For $n < \lambda \leqslant n + p$ and $\alpha = \frac{\lambda - n}{p}$ we have $\mathcal{L}^{p,\lambda}(\Omega) \simeq C^{0,\alpha}(\Omega)$.

Hölder space

The Hölder space $C^{k,\alpha}(\Omega)$ consists of those functions on Ω having continuous derivatives up through order k and such that the kth partial derivatives are Hölder continuous with exponent α, where $0 < \alpha \leqslant 1$. If the Hölder coefficient

$$|u|_{C^{0,\alpha}} = \sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}}$$

is finite, then the function u is said to be Hölder continuous with exponent α.

Theorem

Let $\Phi : [0, R_0] \to [0, +\infty)$ *be non decreasing. Suppose that there exist positive constant* A, B, α, β *with* $\alpha > \beta$ *such that*

$$\Phi(\rho) \leqslant A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \varepsilon\right]\Phi(R) + BR^{\beta}$$

for every $0 < \rho < R \leqslant R_0$. *If there exists* $\varepsilon_0 = \varepsilon_0(\alpha, \beta, A)$ *such that the previous estimate holds for every* $\varepsilon < \varepsilon_0$ *then*

$$\Phi(\rho) \leqslant c(\alpha, \beta, A)\left(\frac{\rho}{R}\right)^{\beta}\left[\Phi(R) + R^{\beta}\right]$$

Since $\Phi(R) \leqslant \Phi(R_0)$ and $R^{\beta} \leqslant R_0^{\beta}$

$$\Phi(\rho) \leqslant c(\alpha, \beta, A, R_0)\left(\frac{\rho}{R}\right)^{\beta}$$

Schauder estimates: constant coefficients case and homogeneous equations

Theorem

Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation

$$\operatorname{div}(\mathcal{A}Du) = 0 \tag{13}$$

with $\mathcal{A}$ an elliptic constant matrix. Then

$$\int_{B_\rho(x_0)} |u|^2\,dx \leqslant c\left(\frac{\rho}{R}\right)^n \int_{B_R(x_0)} |u|^2\,dx \tag{14}$$

and

$$\int_{B_\rho(x_0)} |u - u_\rho|^2\,dx \leqslant c\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} |u - u_R|^2\,dx \tag{15}$$

for every $0 < \rho < R < R_0$ with $B_{R_0} \subset \Omega$.

Non homogeneous equations

Theorem

Let $u \in W^{1,2}(\Omega)$ *be a weak solution of the equation*

$$\text{div}(\mathcal{A}Du) = \text{div}\mathcal{G}$$

with $\mathcal{A}$ *an elliptic constant matrix. Assume*

$$\mathcal{G} \in \mathcal{L}^{2,\lambda}$$

Then

$$Du \in \mathcal{L}^{2,\lambda}$$

for every $0 < \lambda < n+2$.

Hölder regularity- A first Schauder estimate

The Hölder regularity of the datum $\mathcal{G}$ transfers to the gradient of the solution.
Indeed, if $n < \lambda < n + 2$, previous theorem

$$\mathcal{G} \in \mathcal{L}^{2,\lambda} \simeq C^{0,\alpha}$$

$$\Rightarrow Du \in \mathcal{L}^{2,\lambda} \simeq C^{0,\alpha}$$

Non constant coefficients

Theorem

Suppose that $\mathcal{A}(x) \in C^0(\Omega)$ is an elliptic matrix. Let $u \in W^{1,2}(\Omega)$ be a weak solution

$$\operatorname{div}(\mathcal{A}(x)Du) = \operatorname{div}\mathcal{G}$$

If

$$\mathcal{G} \in \mathcal{L}^{2,\mu} \text{ or } \mathcal{G} \in L^{2,\mu} \quad \mu > 0$$

then

$$Du \in L^{2,\lambda}$$

for $0 < \lambda < n$

Morrey regularity not Campanato!

Hölder regularity- A second Schauder estimate

The Hölder regularity of the datum $\mathcal{G}$, in case of continuous coefficients transfer to the solution (not to its gradient). Since

$$\mathcal{G} \in \mathcal{L}^{2,\lambda} \Rightarrow Du \in L^{2,\lambda}$$

we have

$$\int_{B_\rho} |u - u_\rho|^2 \leqslant c\rho^{\lambda+2}$$

Then $u \in \mathcal{L}^{2,\lambda+2} \simeq C^{0,\alpha}$ if $\lambda > n-2$

Schauder Theorem

Theorem

Suppose $\mathcal{A}(x) \in C^{0,\alpha}(\Omega)$ *is an elliptic matrix. Let* $u \in W^{1,2}(\Omega)$ *be a weak solution of the equation*

$$\mathrm{div}(\mathcal{A}(x)Du) = \mathrm{div}\mathcal{G}.$$

If $\mathcal{G} \in C^{0,\beta}$ *then*

$$Du \in C^{0,\gamma}$$

$\gamma = \min\{\alpha, \beta\}$.

The L^p theory

Theorem

Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation

$$\mathrm{div}(\mathcal{A}(x)Du) = \mathrm{div}\mathcal{G}$$

with $\mathcal{A}$ an elliptic constant matrix. Assume that

$$\mathcal{G} \in L^p$$

then

$$Du \in L^p$$

for every $p \geqslant 2$

Theorem

Suppose that $\mathcal{A} \in C^0(\Omega)$ *is an elliptic matrix. Let* $u \in W^{1,2}(\Omega)$ *be a weak solution of the equation*

$$\operatorname{div}(\mathcal{A}(x)Du) = \operatorname{div}\mathcal{G}$$

If

$$\mathcal{G} \in \mathcal{L}^p$$

then

$$Du \in L^p$$

for every $p \geqslant 2$