An Introduction to the Calculus of Variations (with elements of variational inequalities)

Stanislaw Migórski

Jagiellonian University in Krakow, Poland
Faculty of Mathematics and Computer Science

stanislaw.migorski@uj.edu.pl

The name

The name calculus of variations comes from procedures of Lagrange involving an operator δ called a variation.

The calculus of variations broadly interpreted includes all theory and practice concerning the existence and characterization of minima, maxima, and other critical values of a real-valued functional. – G.M. Ewing, *Calculus of Variations with Applications, Dover Publ., New York, 1969.*

The link between Calculus of Variations and Partial Differential Equations has always been strong, because variational problems produce, via their Euler-Lagrange equation, a differential equation and, conversely, a differential equation can often be studied by variational methods.

LECTURES ON THE CALCULUS
OF VARIATIONS

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O. Bolza, Lectures on the Calculus of Variations,
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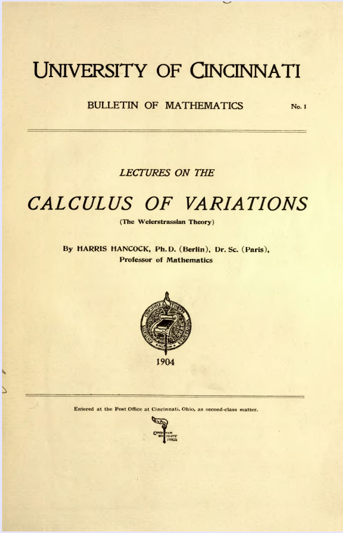
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LECTURES ON THE

CALCULUS OF VARIATIONS

(The Weierstrassian Theory)

By HARRIS HANCOCK, Ph. D. (Berlin), Dr. Sc. (Paris),
Professor of Mathematics

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H. Hancock, Lectures on the Calculus of Variations, (The Weierstrassian Theory), University of Cincinati, 1904.

Two approaches

In the calculus of variations there are, roughly speaking, two ways of proceeding: the classical (indirect) and the direct methods.

Consider minimization problems in $\mathbb{R}^N$.

Let $X \subset \mathbb{R}^N$, $F\colon X \to \mathbb{R}$ and

$$(P) \qquad \inf\{F(x) \mid x \in X\}.$$

The first method consists, if F is continuously differentiable, in finding solutions $x^* \in X$ of

$$F'(x) = 0, \quad x \in X.$$

Then, by analyzing the behavior of the higher derivatives of F, we can determine if x^* is a minimum (global or local), a maximum (global or local) or just a stationary point.

Two methods

The <u>second method</u> consists in considering a minimizing sequence $\{x_n\} \subset X$ so that

$$F(x_n) \to \inf\{F(x) \mid x \in X\}.$$

Then, with appropriate hypotheses on F, we prove that the sequence is compact in X, meaning that

$$x_n \to x^* \in X, \text{ as } n \to \infty.$$

Finally if F is lower semicontinuous, meaning that

$$F(x^*) \leq \liminf_n F(x_n)$$

we have indeed shown that x^* is a minimizer of (P).

Two methods

We can proceed in a similar manner for problems of the calculus of variations.

The first and second methods are then called, respectively, classical and direct methods. However, the problem is now considerably harder because we are working in infinite dimensional spaces.

Classical methods

Recall the basic problem in the calculus of variations

$$(P) \qquad \inf \left\{ I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx \mid x \in X \right\} \to \text{min!}$$

where

$\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a bounded open set,

$u \colon \Omega \to \mathbb{R}^N$, $N \geq 1$ and $\nabla u \in \mathbb{R}^{N\times n}$,

$f \colon \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, $f = f(x, u, \xi)$, is continuous,

X is a space of admissible functions which satisfy $u = u_0$ on $\partial\Omega$, and u_0 is a given function.

Contrary to the case of $\mathbb{R}^N$, we encounter a preliminary problem: what is the best choice for the space X of admissible functions.

Classical methods

A natural choice seems to be $X = C^1(\overline{\Omega})$. There are several reasons, which will be clearer during the study of this area deeper, that indicate that this is not the best choice.

A better choice is the *Sobolev space* $W^{1,p}(\Omega)$, $p \geq 1$. We will say that $u \in W^{1,p}(\Omega)$, if u is (weakly) differentiable and if

$$\|u\|_{W^{1,p}} = \left(\int_\Omega |u(x)|^p + |\nabla u(x)|^p \, dx \right)^{1/p} < \infty.$$

Classical methods

The classical methods introduced by Euler, Hamilton, Hilbert, Jacobi, Lagrange, Legendre, Weierstrass and others.

The most important tool is the *Euler-Lagrange equation*, the equivalent of $F'(x) = 0$ in the finite dimensional case, that should satisfy any $u^* \in C^2(\overline{\Omega})$ minimizer of (P), namely (we write here the equation in the case $N = 1$)

$$(E) \qquad \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_{\xi_i}(x, u^*, \nabla u^*) = f_u(x, u^*, \nabla u^*) \text{ for all } x \in \overline{\Omega},$$

where $f_{\xi_i} = \partial f / \partial \xi_i$ and $f_u = \partial f / \partial u$.

A solution u^* of (E) is called sometimes a *stationary point* of I.

Classical methods. Example

This is the most celebrated problem of the calculus of variations.

In the case of the Dirichlet integral

$$(P) \qquad \inf\left\{ I(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx \mid u = u_0 \text{ on } \partial\Omega \right\}.$$

The Euler-Lagrange equation reduces to *Laplace equation*, i.e., $\Delta u^* = 0$.

Classical methods

Comments:

In general, finding a C^2 solution of (E) is a difficult task, unless perhaps $n = 1$ or the equation (E) is linear.

We need to know if a solution u^* of (E) is, in fact, a minimizer of (P). If $(u, \xi) \mapsto f(x, u, \xi)$ is convex for every $x \in \Omega$, then u^* is indeed a minimum of (P); this happens for the Dirichlet integral.

If, however, $(u, \xi) \mapsto f(x, u, \xi)$ is not convex, several criteria, specially in the case $n = 1$, can be used to determine the nature of the stationary point. Such criteria are for example, Legendre, Weierstrass, Weierstrass-Erdmann, Jacobi conditions and other theories.

Classical methods

Comments:
If $\Omega = [a, b]$, i.e., $n = 1$, then it is possible to show that any minimizer u^* of (P) satisfies a different form of the Euler-Lagrange equation: for all $x \in [a, b]$, we have

$$\frac{d}{dx}\Big(f(x, u^*(x), u^{*'}(x)) - u^{*'}(x) f_\xi(x, u^*(x), u^{*'}(x))\Big) = f_x(x, u^*(x), u^{*'}(x)).$$

This form is particularly useful when f does not depend explicitely on the variable x, because then a first integral of (E) is of the form

$$f(u^*(x), u^{*'}(x)) - u^{*'}(x)\, f_\xi(u^*(x), u^{*'}) = \mathrm{const.} \quad \text{for all} \quad x \in [a, b].$$

Classical methods

Another idea is that the solutions to (E) are also solutions (and conversely) of

$$\begin{cases} u'(x) = H_v(x, u(x), v(x)) \\ v'(x) = -H_u(x, u(x), v(x)), \end{cases}$$

where $v(x) = f_\xi(x, u(x), u'(x))$ and H is the Legendre transform of f, i.e.,

$$H(x, u, v) = \sup_{\xi \in \mathbb{R}} \{ v\xi - f(x, u, \xi) \}$$

In classical mechanics f is called the *Lagrangian* and H the *Hamiltonian*.

Direct methods

The direct methods were introduced by Hilbert, Lebesgue and Tonelli.

The idea is to split the problem into two pieces: existence of minimizers in Sobolev spaces and then regularity of the solution.

For instance, the existence of minimizers of (P) in Sobolev spaces is established in $W^{1,p}(\Omega)$. We can see that, sometimes, minimizers of (P) are more regular than in a Sobolev space, they are in C^1 or even in C^∞, if the data Ω, f and u_0 are sufficiently regular.

What is the idea of the direct method to show existence of minimizers in Sobolev spaces?

Existence by the direct methods

As in the finite dimensional case, we start by considering a minimizing sequence $\{u_n\} \subset W^{1,p}(\Omega)$, which means that

$$I(u_n) \to \inf\{I(u) \mid u = u_0 \text{ on } \partial\Omega \text{ and } u \in W^{1,p}(\Omega)\} = \min!, \text{ as } n \to \infty.$$

The first step consists in showing that the sequence $\{u_n\}$ is compact, i.e., that the sequence converges to an element $u^* \in W^{1,p}(\Omega)$. This, clearly, depends on the topology that we have on $W^{1,p}(\Omega)$. The natural one is the one induced by the norm, that we call *strong convergence*

$$u_n \to u^* \ \text{ in } \ W^{1,p}(\Omega).$$

However, it is, in general, not an easy matter to show that the sequence converges in such a strong topology. It is often better to weaken the notion of convergence and to consider the so called *weak convergence*

$$u_n \to u^* \ \text{ weakly in } \ W^{1,p}(\Omega).$$

Direct methods

To obtain the weak convergence is much easier and it is enough, for example if $p > 1$, to show (up to the extraction of a subsequence) that

$$\|u_n\|_{W^{1,p}(\Omega)} \leq c,$$

where $c > 0$ is a constant independent of n. Such an estimate follows, for instance, if we impose a coercivity assumption on the function f of the type

$$\lim_{|\xi|\to\infty} \frac{f(x,u,\xi)}{|\xi|} = +\infty \text{ for all } (x,u) \in \overline{\Omega} \times \mathbb{R}.$$

We observe that the Dirichlet integral, with $f(x,u,\xi) = |\xi|^2/2$, satisfies this hypothesis.

Direct methods

The second step in the direct methods consists in showing that the functional I is lower semicontinuous with respect to weak convergence, namely

$$u_n \to u^* \;\; \text{weakly in} \;\; W^{1,p}(\Omega) \quad \Longrightarrow \quad I(u^*) \leq \liminf_n I(u_n).$$

This conclusion is true if

$$\xi \mapsto f(x, u, \xi) \;\; \text{is convex} \;\; (x, u) \in \overline{\Omega} \times \mathbb{R}.$$

Since $\{u_n\}$ was a minimizing sequence, we deduce that u^* is indeed a minimizer of (P).

Regularity

All regularity results are obtained about solutions of the Euler-Lagrange equation and therefore not only minimizers of (P).

We quote: Agmon, Bernstein, Calderon, De Giorgi, Douglis, E. Hopf, Leray, Liechtenstein, Morrey, Moser, Nash, Nirenberg, Rado, Schauder, Tonelli, Weyl and Zygmund.

Direct methods

The following theorem was first established by De Giorgi, then simplified by Moser and also proved, independently but at the same time, by Nash.

Theorem (1956/57) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $v \in W^{1,2}(\Omega)$ be a solution of

$$\sum_{i,j=1}^{n} \int_{\Omega} \left(a_{ij}(x) v_{x_i}(x) \varphi_{x_j}(x) \right) \, dx = 0 \quad \text{for all} \quad \varphi \in v \in W_0^{1,2}(\Omega),$$

where $a_{ij} \in L^\infty(\Omega)$ are such that

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \gamma \|\xi\|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n, \ \gamma > 0.$$

Then there exists $0 < \alpha < 1$ such that $v \in C^{0,\alpha}(D)$ for every $D \subset \overline{D} \subset \Omega$.

Equivalent problems

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $M = \{v \in C^1(\overline{\Omega}) \mid v = 0 \text{ on } \partial\Omega\}$.

(A) Variational problem

$$\int_\Omega \left(\frac{1}{2} \sum_{i=1}^n (D_i u)^2 - fu \right) dx \to \min!, \quad u \in C^1(\overline{\Omega}), \quad u = g \ \text{ on } \partial\Omega$$

(B) Generalized BVP

$$\int_\Omega \left(\sum_{i=1}^n D_i u \, D_i v - fv \right) dx = 0, \quad \text{for all } v \in M, \quad u = g \ \text{ on } \partial\Omega$$

(C) BVP

$$-\Delta u = f \ \text{ in } \Omega, \quad u = g \ \text{ on } \partial\Omega \ \text{ (the Euler equation for (A))}$$

Equivalent problems

Theorem

Let $f \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$. Then

(a) *for $u \in C^2(\overline{\Omega})$: (A), (B), and (C) are equivalent,*

(b) *for $u \in C^1(\overline{\Omega})$: (A) and (B) are equivalent.*

Proof: uses a typical reduction to extremal problem for real functions

Let

$$F(u) = \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{n} (D_i u)^2 - fu \right) dx$$

and

$$\varphi(t) := F(u + tv) \ \text{ for } t \in \mathbb{R}, \ \text{fixed } v \in M, \ \ (\varphi \text{ depends on } v).$$

Equivalent problems

Let $u \in C^2(\overline{\Omega})$.

$(A) \Longrightarrow (B)$.

Let u solve (A), i.e., $F(u) = \min!$, $u = g$ on $\partial\Omega$ and $u \in C^1(\overline{\Omega})$.
If $v \in M$, $t \in \mathbb{R}$, then $u + tv \in C^1(\overline{\Omega})$ and $u + tv = g$ on $\partial\Omega$.
Hence, the function φ has a minimum at $t = 0$, so $\varphi'(0) = 0$. This implies

$$\int_\Omega \left(\sum_{i=1}^n D_i u \, D_i v - fv \right) dx = 0, \quad \text{for all } v \in M, \quad u = g \quad \text{on } \partial\Omega,$$

which is equivalent to (B).

Equivalent problems

$(B) \Longrightarrow (C)$.

Let u solve (B), i.e.,

$$\int_\Omega \left(\sum_{i=1}^{n} D_i u \, D_i v - fv \right) dx = 0, \quad \text{for all } v \in M.$$

By intergation by parts, we have

$$\int_\Omega (-\Delta u - f) \, v \, dx = 0, \quad \text{for all } v \in M$$

and in particular for all $v \in C_0^\infty(\Omega)$. By the variational lemma, we get

$$-\Delta u - f = 0.$$

Since $u = g$ on $\partial\Omega$, u solves (C).

Equivalent problems

$(C) \Longrightarrow (B)$.

We multiply

$$-\Delta u = f$$

by $v \in M$ and integrate on Ω.
Integration by parts implies that u solves (B).

Equivalent problems

$(A) \iff (B)$.

Since $\varphi\colon \mathbb{R} \to \mathbb{R}$ is quadratic, we have

$$\text{“}\varphi'(0) = 0\text{”} \quad \iff \quad \text{“}\varphi \text{ has a minimum at } t = 0\text{”}.$$

Hence, (A) is equivalent to "$\varphi'(0) = 0$ for all $v \in M$" which is equivalent to (B).

Proof of (b):
(A) and (B) are equivalent, we have only used that $u \in C^1(\overline{\Omega})$.

Variations

Remark:

$$\delta^k F(u; v) := \varphi^{(k)}(0)$$

is called kth variation of F at the point u in the direction v. In particular

$$\delta F(u; v) = \int_\Omega \left(\sum_{i=1}^n D_i u \, D_i v - fv \right) \, dx,$$

$$\delta^2 F(u; v) = \int_\Omega \sum_{i=1}^n (D_i v)^2 \, dx.$$

Note: $(B) \iff$ the first variation vanishes: $\delta F(u; v) = 0$ for all $v \in M$.

Variational inequalities

Elliptic variational inequalities (EVIs)

Let X be a Hilbert space (extensions are possible) with $\langle \cdot, \cdot \rangle$, $A\colon X \to X$ be an operator, $K \subset X$, $K \neq \emptyset$, and $j\colon X \to \overline{\mathbb{R}}$ be a proper functional.

Elliptic variational inequality of the first kind:

$$(*) \qquad \begin{cases} \text{given } f \in X, \text{ find } u \in K \text{ such that} \\ \langle Au, v-u \rangle \geq \langle f, v-u \rangle \ \text{ for all } \ v \in K. \end{cases}$$

Elliptic variational inequality of the second kind:

$$(**) \qquad \begin{cases} \text{given } f \in X, \text{ find } u \in X \text{ such that} \\ \langle Au, v-u \rangle + j(v) - j(u) \geq \langle f, v-u \rangle \ \text{ for all } \ v \in X. \end{cases}$$

If we take $j = I_K$, the indicator function of the set K, then the variational inequality of second kind reduces to the variational inequality of the first kind.

Elliptic variational inequality $(**)$

Theorem (Well posedness)

Let X be a Hilbert space, and $A\colon X \to X$ satisfy

$$\begin{cases} \|Au - Av\| \le M\|u - v\| \text{ (Lipschitz continuous } M > 0\text{)}, \\ \langle Au - Av, u - v\rangle \ge m\|u - v\|^2 \text{ (strongly monotone } m > 0\text{)}, \end{cases}$$

and $j\colon X \to \overline{\mathbb{R}}$ be a proper, convex and l.s.c., $f \in X$.

Then $(**)$ has a unique solution $u \in X$ and

$$\|u_1 - u_2\| \le \frac{1}{m}\|f_1 - f_2\|, \text{ where } u_i = u(f_i), \ i = 1, 2.$$

A particular case of $(**)$

Corollary (Operator $A = \mathrm{Id}$)

Let X be a Hilbert space, and $j\colon X \to \overline{\mathbb{R}}$ be proper, convex and l.s.c. Then, for any $f \in X$, there exists a unique solution $u \in X$ to

$$(***) \qquad \langle u, v-u\rangle + j(v) - j(u) \geq \langle f, v-u\rangle \quad \text{for all} \quad v \in X,$$

and $f \mapsto u$ is Lipschitz continuous.

The unique solution $u \in X$ to $(***)$ is called the proximal element of f with respect to j.
The operator $\mathrm{Prox}_j\colon X \to X$, $\mathrm{Prox}_j(f) = u$ is called the proximity operator (J. Moreau, 1965). We know that

$$\|\mathrm{Prox}_j(f_1) - \mathrm{Prox}_j(f_2)\| \leq \|f_1 - f_2\| \quad \text{(non-expansive operator)},$$
$$\langle \mathrm{Prox}_j(f_1) - \mathrm{Prox}_j(f_2), f_1 - f_2\rangle \geq 0 \quad \text{(monotone operator)}.$$

A particular case: the variational equality

Corollary (Lax-Milgram lemma)

Let X be a Hilbert space,

$$a\colon X \times X \to \mathbb{R} \text{ is bilinear, bounded, coercive}$$

$$(a(v,v) \geq m\|v\|^2 \text{ for all } v \in X \text{ with } m > 0),$$

and

$$l \in X^*.$$

Then, there exists a unique solution $u \in X$ to

$$a(u,v) = l(v) \quad \text{for all} \quad v \in X,$$

Equivalent formulation

Theorem (Energy formulation)

Let X be a Hilbert space,

$a\colon X \times X \to \mathbb{R}$ *is bilinear, bounded, coercive, symmetric,*

$j\colon X \to \overline{\mathbb{R}}$ *be proper, convex and l.s.c.,*

and $l \in X^*$.

Then,

$u \in X$ *solves* $a(u, v - u) + j(v) - j(u) \geq l(v - u)$ *for all* $v \in X$

$\Longleftrightarrow$

$u \in X$ *solves* $\min_{v \in X} J(v)$,

where

$$\begin{cases} J\colon X \to \overline{\mathbb{R}} \\ J(v) := \frac{1}{2} a(v, v) + j(v) - l(v) \quad \text{for} \quad v \in X. \end{cases}$$

Another equivalent formulation

Theorem (Differentiable potential)

Let X be a Hilbert space,

$a\colon X \times X \to \mathbb{R}$ be bilinear, bounded, coercive, symmetric,

$j\colon X \to \mathbb{R}$ be convex and l.s.c. and Gâteaux differentiable (finite),

and $A\colon X \to X$ is such that $\langle Au, v\rangle = a(u, v)$.

Then,

$$u \in X \text{ solves } a(u, v-u) + j(v) - j(u) \geq l(v-u) \text{ for all } v \in X$$

$$\Longleftrightarrow$$

$$u \in X \text{ solves } Au + \nabla j(u) = f.$$

Recall: Let $\varphi\colon X \to \mathbb{R}$, $u \in X$, X be a Hilbert space. Then φ is "Gâteaux differentiable" at u if and only if

$$\exists\, \nabla\varphi(u) \in X \text{ such that } \lim_{t\to 0} \frac{\varphi(u+tv) - \varphi(u)}{t} = \langle \nabla\varphi(u), v\rangle \text{ for } v \in X.$$

Elliptic quasivariational inequality

Other classes of inequalities can be considered:

(1) find $u \in X$ such that

$$a(u, v-u) + j(u, v) - j(u, u) \geq \langle f, v-u \rangle \quad \text{for all} \quad v \in X,$$

(2) find $u \in K(u)$ such that

$$a(u, v-u) \geq \langle f, v-u \rangle \quad \text{for all} \quad v \in K(u).$$

Minty formulation

Theorem (Minty inequality, 1961)

Let X be a Hilbert space,
$a\colon X \times X \to \mathbb{R}$ be bilinear, bounded, positive,
K be nonempty, closed, convex.
Then, the following problems are equivalent

$$(1)\quad \text{find } \ u \in K \ \text{ such that}$$
$$a(u, v-u) \geq \langle f, v-u\rangle \quad \text{for all } \ v \in K,$$

$$(2)\quad \text{find } \ u \in K \ \text{ such that}$$
$$a(v, v-u) \geq \langle f, v-u\rangle \quad \text{for all } \ v \in K.$$

Generalized Minty formulation

Theorem (Minty inequality)

Let V be a reflexive Banach space, $f \in V^*$
$A\colon V \to V^*$ be monotone and hemicontinuous,
$K \subset V$ be nonempty, closed, convex.

Then, the following problems are equivalent

$$(1) \quad \text{find} \quad u \in K \quad \text{such that} \quad \langle Au - f, v - u \rangle \geq 0 \quad \text{for all} \quad v \in K,$$

$$(2) \quad \text{find} \quad u \in K \quad \text{such that} \quad \langle Av - f, v - u \rangle \geq 0 \quad \text{for all} \quad v \in K.$$

For the proof (1) $\Longrightarrow$ (2), the monotonicity is needed.

For the proof (2) $\Longrightarrow$ (1), the hemicontinuity is needed.

A saddle point formulation

Consider $L\colon X \times X \to \mathbb{R}$ defined by

$$L(v, w) := a(v, v-w) - \langle f, v-w \rangle \quad \text{for} \quad v,\, w \in X.$$

$\Longrightarrow$: if $u \in K$ solves (1) (or equivalently the Minty form (2)), then

$$L(u, v) \le L(u, u) = 0 \le L(v, u) \quad \text{for all} \quad v \in K, \tag{SP}$$

i.e., the pair $(u, u) \in K \times K$ is a saddle point of L on $K \times K$.

$\Longleftarrow$: if $(u, \widetilde{u}) \in K \times K$ is a saddle point of L on $K \times K$, then $u = \widetilde{u}$ is a solution to (1) (or (2)).

Conclusion:

$$u \in K \text{ solves (1)} \iff (u, u) \text{ is a saddle point of } L \text{ on } K \times K.$$

Note that (SP) implies

$$L(u, u) = \max_{z \in K} L(u, z) \quad \text{and} \quad L(u, u) = \min_{w \in K} L(w, u).$$

Penalty (penalization) method

Let X be a Hilbert space. Consider the EVI of first kind

$$(*) \qquad \begin{cases} \text{given } f \in X, \text{ find } u \in K \text{ such that} \\ \langle Au, v-u\rangle \geq \langle f, v-u\rangle \ \text{ for all } \ v \in K. \end{cases}$$

Goal:

- give another proof of existence and uniqueness to $(*)$,
- provide an approximation scheme to $(*)$,
- show the strong convergence result.

Penalty method: hypotheses

For the EVI in $(*)$, we need the hypotheses.

$$A\colon X \to X \text{ is strongly monotone and Lipschitz.} \qquad H(A)$$

$$K \subset X \;\text{ is nonempty, closed, convex.} \qquad H(K)$$

$$\begin{cases} P\colon X \to X \text{ is monotone and Lipschitz,} \\ \langle Pu, v-u\rangle \le 0 \;\text{ for all }\; u \in X,\, v \in K, \\ Pu = 0 \;\text{ if and only of }\; u \in K. \end{cases} \qquad H(P)$$

Operator P in $H(P)$ is called a penalty operator. Such operator always exists for K as in $H(K)$.

Penalty method: main result

Recall the variational inequality

$$\text{find } u \in K \text{ such that } \langle Au, v-u\rangle \geq \langle f, v-u\rangle \text{ for all } v \in K. \qquad (*)$$

Consider, for any $\lambda > 0$, the problem:

$$\text{find } u_\lambda \in X \text{ such that } Au_\lambda + \frac{1}{\lambda}Pu_\lambda = f. \qquad (P_\lambda)$$

Theorem (Penalty)

Let X be a Hilbert space, $H(A)$, $H(K)$, $H(P)$ hold, and $f \in X$. Then,

(a) *for any $\lambda > 0$, there is the unique $u_\lambda \in X$ solution to (P_λ),*
(b) *there exists the unique solution $u \in K$ solution to $(*)$,*
(c) *$u_\lambda \to u$ in X, as $\lambda \to 0$.*

Penalty operator

Lemma (Projection lemma)

Let X be a Hilbert space, $K \subset X$ be nonempty, closed, convex. Then, for any $f \in X$, there is the unique $u \in K$ such that

$$\|u - f\| = \min_{v \in K} \|v - f\|.$$

The element $u \in K$ is called the projection of f on K. The operator $\pi_K \colon X \to X$ is called the projection operator on K. We write $u = \pi_K(f)$. Moreover

$$u = \pi_K(f) \iff u \in K,\ \langle u - f, v - u \rangle \geq 0 \ \text{ for all } \ v \in K.$$

Next, we define $P \colon X \to X$ by $P(u) = (I - \pi_K)u$ for $u \in X$.

Then P is the penalty operator since it satisfies $H(P)$.

Stability with respect to (A, f, K)

Theorem (Convergence)

Let X be a Hilbert space, $A\colon X \to X$ be strongly monotone and Lipschitz, $K \subset X$ be nonempty, closed, convex, $f \in X$.

For all $n \in \mathbb{N}$, let $A_n\colon X \to X$ be strongly monotone and Lipschitz, $K_n \subset X$ be nonempty, closed, convex, $f_n \in X$.
Assume

$$f_n \to f \text{ in } X,$$
$$A_n v_n \to Av \text{ for any } \{v_n\} \subset K_n,\ v_n \to v \text{ in } X,\ v \in K,$$
$$K_n \xrightarrow{M} K.$$

Let

$$u \in K \text{ be the unique solution to } \langle Au - f, v - u\rangle \geq 0 \text{ for all } v \in K,$$
$$u_n \in K_n \text{ be the unique solution to } \langle A_n u_n - f_n, v - u_n\rangle \geq 0 \text{ for all } v \in K_n.$$

Then $u_n \to u$ in X.

Convergence of sets, Mosco (1969)

Given a Banach space X, a sequence $\{K_n\}$ of closed and convex sets in X, is said to converge to a closed and convex set $K \subset X$ in the Mosco sense, denoted by

$$K_n \xrightarrow{M} K$$

as $n \to \infty$, iff

$$s\text{-}\liminf K_n = w\text{-}\limsup K_n = K,$$

and iff

(m_1) for any $z_n \in K_n$, $z_n \rightharpoonup z$ in X, up to a subsequence, we have $z \in K$,

(m_2) for any $z \in K$, there exists $z_n \in K_n$ with $z_n \to z$ in X.

Given (X, τ), we recall the Kuratowski limits of sets $\{A_n\} \subset X$:

$$\tau\text{-}\liminf A_n = \{x \in X \mid x = \tau\text{-}\lim x_n, x_n \in A_n \text{ for all } n \geq 1\},$$

$$\tau\text{-}\limsup A_n = \{x \in X \mid x = \tau\text{-}\lim x_{n_k}, x_{n_k} \in A_{n_k},\ n_1 < n_2 < \ldots < n_k < \ldots\}$$

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Thank you very much for your attention!