Politechnika Łódzka

Analysis in Tatra
Seminar for Students

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Twierdzenie Bolzano i co dalej?

Twierdzenie Poincaré-Mirandy i jego uogólnienia

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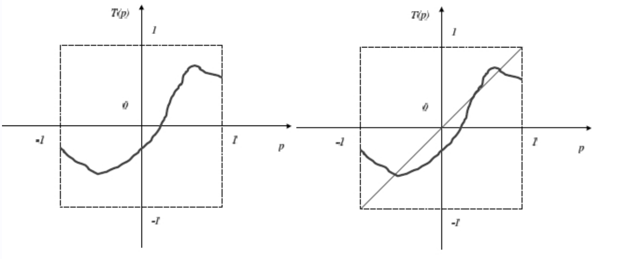
Some history

▲ Bernard Bolzano (1791-1848): a continuous function $f : K = [a, b] \to \mathbb{R}$ such that $f(a)f(b) \leqslant 0$ equals zero at some $\bar{x} \in K$.



Theorem.jpg

Theorem (Bolzano Fixed Point)

If $f(a) \geqslant a$, $f(b) \leqslant b$ (this holds $\Leftrightarrow$ (INWD) $f(y) \in y + T_K(y)$ for all $y \in K$), then f has a fixed point $\bar{x} \in K$, i.e. $f(\bar{x}) = \bar{x}$.

Corollary

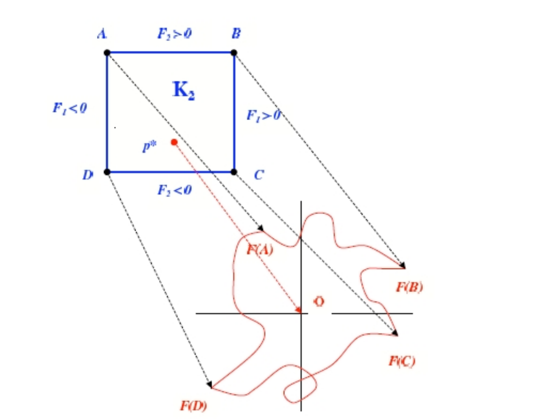
If $a \leqslant 0 \leqslant b$ and $f(a) \geqslant 0$, $f(b) \leqslant 0$ (this holds $\Leftrightarrow$ (T) $f(y) \in T_K(y)$ for all $y \in K$), then f has a fixed point.

In 1883, Henri Poincaré announced (verbatim translation by F. Browder):

▲ *Let $F_1, ..., F_n$ be continuous functions of n variables $x_1, ..., x_n$: the variable x_i is subjected to vary between the limits $-a_i$ and $+a_i$. Let us assume that:*

- *for $x_i = a_i$, F_i is constantly positive*;
- *for $x_i = -a_i$, F_i is constantly negative*;

I say: *there will exist a system of values of x where all f_i vanish.*



Poincaré concluded with a hint to proof in „an important theorem of Leopold Kronecker" from 1869. In 1886 Poincaré gave his famous paper on the homotopy invariance of the index: a basis of a modern proof; but the result was rapidly forgotten.

The result was implicitly rediscovered in 1911 by L. E. J. Brouwer who proved that:
▲ Under a continuous map of the unit cube into itself which displaces every point less then half unit, the image has an interior point.

Brouwer's fixed point theorem, $n = 3$, was proved by in 1909; an equivalent was established by P. Bohl in 1904; the proof for arbitrary n is due to J. H. Hadamard in 1910 (Kronecker's index). In 1912 Brouwer proved it with simplicial approximations and inceptions of degree theory.

▲ Rediscovered in 1940 by Sivio Cinquini (incorrect proof);
▲ Proved by Carlo Miranda in 1941 (showing the equivalence with the Brouwer fixed point theorem).

Theorem (Poincaré-Miranda)

Let $Q = \Pi_{k=1}^{n}[a_k, b_k]$ *be an n-dimensional cube and let*

$$F_k^- := \{x \in Q \mid x_k = a_k\}, \quad F_k^+ := \{x \in Q \mid x_k = b_k\}, \quad k = 1, 2, ..., n.$$

Let $f = (f_1, ..., f_n) : Q \to \mathbb{R}^n$ *be continuous and for all* $k = 1, ..., n$

$$(-T) \quad f_k(x) \begin{cases} \leqslant 0 & \text{for every } x \in F_k^- \\ \geqslant 0 & \text{for every } x \in F_k^+ \end{cases} \quad \text{or} \quad (T) \quad f_k(x) \begin{cases} \geqslant 0 & \text{for every } x \in F_k^- \\ \leqslant 0 & \text{for every } x \in F_k^+. \end{cases}$$

Then there is $\bar{x} \in Q$ *such that* $f(\bar{x}) = 0$.

The assertion holds true if $(-T)$ *and* (T) *are „mixed":*
$(mixT)$ *if* $x \in F_k^-$ *and* $y \in F_k^+$*, then* $f_k(x) \cdot f_k(y) \leqslant 0$.

The **proof** is simple. One shows that if $f(x) \neq 0$ on ∂Q, then f is homotopic to $-I$; hence $\deg(f, \operatorname{int} Q) = (-1)^n \neq 0$ (last proof: M Vrahatis 1999 in PAMS)

Corollary (Zgliczyński (2001))

If $0 \in Q$ and for all $k = 1, \dots, n$, (T) holds, i.e.,

$$(T) \qquad f_k(x) \geqslant 0 \ \text{ for } \ x \in F_k^- \ \text{ and } \ f_k(y) \leqslant 0 \ \text{ for } \ y \in F_k^+,$$

then there is $x^* \in Q$ such that $f(x^*) = x^*$.

Condition (T) cannot be replaced by $(-T)$ and the result is not true when $0 \notin Q$.

Theorem (Ghezzo (1947), Schäfer (2007), Mawhin (2013))

(1) Let $\mathbb{Q} = \{x \in \ell^2 \mid |x_k| \leqslant \frac{1}{k}\}$ be the Hilbert cube and $f : \mathbb{Q} \to \ell^2$ and such that for all $k \in \mathbb{N}$

$$(mixT) \qquad f_k(x_1, \dots, x_{k-1}, -\frac{1}{k}, x_{k+1}, \dots) \cdot f_k(x_1, \dots, x_{k-1}, \frac{1}{k}, x_{k+1}, \dots) \leqslant 0,$$

then f has a zero.

(2) If for all $k \in \mathbb{N}$,

$$(T) \qquad f_k(x_1, \dots, x_{k-1}, -\frac{1}{k}, x_{k+1}, \dots) \geqslant 0, \ \ f_k(x_1, \dots, x_{k-1}, \frac{1}{k}, x_{k+1}, \dots) \leqslant 0,$$

then f has a fixed point and a zero.

Corollary (W. Hurewicz)

Assume that for any $k = 1, \dots, n$, there are a nonempty closed set $B_k \subset Q$, open disjoint sets $U_k^-, U_k^+ \subset Q \setminus B_k$ such that:

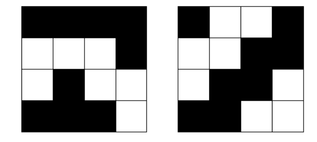
$$Q \setminus B_k = U_k^- \cup U_k^+$$

and

$$F_k^- \subset U_k^-, \quad F_k^+ \subset U_k^+.$$

Then $\bigcap_{k=1}^n B_k \neq \emptyset$ (there is ℓ^2 version, too).

For proof: let $f_k(x) = \eta_k(x) d(x, B_k)$, where $\eta_k = -1$ on U_k, $+1$ on U^k and 0 on B_k, $k = 1, \dots, n$.



Theorem (H. Steinhaus)

Consider an $n \times n$ chessboard and place mines on any set of squares. Then: either a king can move from the left to the right omitting mines or a rook can move from the bottom to the top using only mined squares.

Theorem (Browder, Halpern-Benjamin, Deimling (1989))

Let E be a Banach space. If $K \subset E$ is closed convex, $f : K \to E$ is compact and for any $y \in K$

(*INWD*) $$f(y) \in y + T_K(y),$$

where

$$T_K(y) = \overline{\bigcup_{h>0} \frac{K - y}{h}},$$

, then f has a fixed point.

For (a simple) **proof** assume that E is a Hilbert space. Let $r : E \to K$ be a (metric) retraction i.e., $\|x - r(x)\| = d(x, K)$, $x \in K$. Let $g = f \circ r$; then $g : E \to E$ is compact and, by the Schauder theorem, has a fixed point $\bar{x} \in E$

$$\bar{x} = g(\bar{x}) = f(\bar{y}) \text{ where } \bar{y} = r(\bar{x}).$$

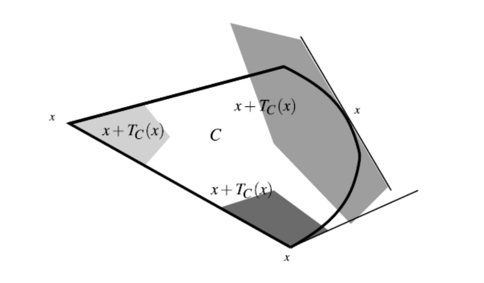
For each $z \in K$, $\langle \bar{x} - \bar{y}, z - \bar{y} \rangle \leqslant 0$, i.e.

$$K - \bar{y} \subset \{v \in E \mid \langle \bar{x} - \bar{y}, v \rangle \leqslant 0\}.$$

Hence if $v \in T_K(\bar{y})$ then $\langle \bar{x} - \bar{y}, v \rangle \leqslant 0$.
Thus $v = \bar{x} - \bar{y} = f(\bar{y}) - \bar{y} \in T_K(\bar{y}) \implies \|\bar{x} - \bar{y}\|^2 = 0$.

$K \subset E$ closed convex, $x \in K$,

$$T_K(x) = \overline{\bigcup_{h>0} \frac{K-x}{h}}$$



Cone.jpg

In fact it is a wedge, i.e. convex and if $v \in T_K(x)$, $\lambda \geqslant 0$, then $\lambda v \in T_K(x)$.

- *If* $x \in \operatorname{int} K$, *then* $T_K(x) = E$.
- $v \in T_K(x)$ *if and only if there are sequences* (v_n), (h_n) *such that* $v = \lim_{n\to\infty} v_n$, $h_n \searrow 0$ *and* $x + h_n v_n \in K$.

Example

(1) If $K = D(0,R)$ (a closed ball) in a Hilbert space E, then for $x \in H$ with $\|x\| = R$

$$T_K(x) = \{v \in E \mid \langle x, v\rangle \leqslant 0\}.$$

(2) If $Q = \Pi_{k=1}^n [a_k, b_k]$ is a cube and $x \in \partial Q$, then

$$v = (v_1, ..., v_n) \in T_Q(x) \iff \forall\, k = 1, ..., n \quad v_k \text{ is } \begin{cases} \geqslant 0 & \text{if } x_k = a_k \\ \leqslant 0 & \text{if } x_k = b_k. \end{cases}$$

(3) If $K = [a, b]$, then $T_K(y) = \mathbb{R}$ if $y \in (a, b)$, $T_K(a) = [0, +\infty)$, $T_K(b) = (-\infty, 0]$.

(4) (Aubin-Frankowska) Let $D \subset \mathbb{R}^n$ closed convex. If $K = \{u \in L^p(\Omega, \mathbb{R}^n) \mid u(x) \in D \text{ for a.a } x \in \Omega\}$, $u \in K$, then K is closed and convex in L^p and

$$v \in T_K(u) \iff v(x) \in T_D(u(x)) \text{ for a.a. } x \in \Omega.$$

Corollary

A map $f : Q \to \mathbb{R}^n$

▲ *satisfies* $(-T)$ *of the Miranda theorem* $\iff \forall\, x \in K \quad -f(x) \in T_Q(x)$.

▲ *satisfies* (T) *of the Miranda theorem* $\iff \forall\, x \in K \quad f(x) \in T_Q(x)$.

▲ *A map* $f : \mathbb{Q} \to \ell^2$ *satisfies the assumption of the infinite dimensional Miranda theorem* $\iff \forall\, x \in \mathbb{Q}\;\; f(x) \in T_{\mathbb{Q}}(x)$.

Theorem (Halpern (1965), Halpern-Bergman (1968), Browder (1968))

- *If* $K \subset \mathbb{E}$ *is compact convex,* $f : K \to \mathbb{E}$ *is continuous and tangent, i.e.*

$$\forall\, x \in K \;\; f(x) \in T_K(x),$$

then f *has a zero.*

- *If* f *is inward, i.e.,* $\forall\, x \in K \;\; f(x) \in x + T_K(x)$ *or outward, i.e.* $\forall\, x \in K \;\; f(x) \in x - T_K(x)$, *then* f *has a fixed point.*

Remark

If $0 \in K$, then $T_K(x) \subset x + T_K(x)$. Hence tangency (condition (T) in Miranda) implies inwardness and, hence, fixed points (and $(-T)$ does not).

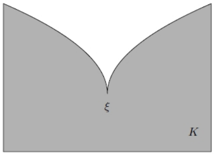
We are going to formulate results that generalize the above „Miranda" theorems.

Definition (WK (1997))

A closed set $K \subset \mathbb{E}$ is an $\mathcal{L}$-retract if there is a neighborhood retraction $r : U \to K$ such that $\|r(x) - x\| \leqslant Ld(x, K)$ for some $L \geqslant 1$.

Example

K is an $\mathcal{L}$-retract if:
(1) K is closed convex (with $L = 1 + \varepsilon$; with $L = 1$ if $\mathbb{E}$ is Hilbert);
(2) K is epi-Lipschitz (locally the epigraph of a Lipschitz functional);
(3) K is locally convex closed subset of a Riemannian manifold.



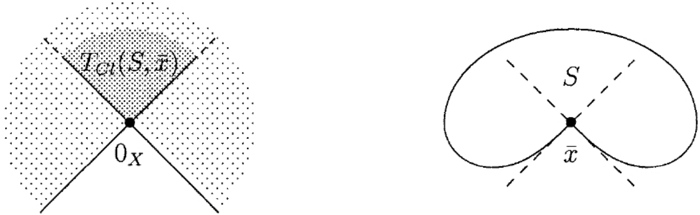
The set K is not an L-retract, because of the "cusp" at ξ.

Definition (Clarke)

Let $K \subset \mathbb{E}$ be closed and $x \in K$. The Clarke tangent cone

$$C_K(x) := \{v \in \mathbb{E} \mid \lim_{h \to 0^+,\, y \to x,\, y \in K} \frac{d(y + hv, K)}{h} = 0\}.$$

- $v \in C_K(x) \iff$ if $x_n \to x$, $h_n \to 0^+ \Rightarrow \exists\, v_n \to v$ such that $x_n + h_n v_n \in K$.
- If K is closed convex then $C_K(x) = T_K(x)$.



tangent.jpg

Theorem (H. Ben-El-Mechaiekh, WK (TAMS 1999))

Let K be a compact $\mathcal{L}$-retract, the Euler characteristic $\chi(K) \neq 0$. Let $f : K \to \mathbb{E}$ be continuous and tangent:

$$\forall\, x \in K \;\; f(x) \in C_K(x),$$

then there is $\bar{x} \in K$ with $f(\bar{x}) = 0$ (fixed point version is true, too).

Definition

If $K \subset \mathbb{E}$ closed, $x \in K$, then the normal cone to K at x

$$N_K(x) := \{p \in \mathbb{E}^* \mid \forall\, v \in C_K(x) \;\; \langle p, v\rangle \leqslant 0\}.$$

▲ If K is closed convex, then

$$N_K(x) := \{p \in \mathbb{E}^* \mid \max_{y \in K}\langle p, y\rangle = \langle p, x\rangle\} = \{p \in \mathbb{E}^* \mid \forall\, v \in T_K(x) \;\; \langle p, v\rangle \leqslant 0\}.$$

Theorem (A. Ćwiszewski, WK (NA 2005))

If K is as above, $\Phi : K \to \mathbb{E}^*$ is continuous (typically $\Phi(x) = \nabla F$), then there is a generalized critical point, i.e., $\bar{x} \in K$ such that

$$\Phi(\bar{x}) \in N_K(\bar{x}).$$

Compactness removed

Relaxing compactness of the domain requires compactness in the mapping.

Theorem (K. Deimling (NA 1992))

Let $K \subset \mathbb{E}$ be closed bounded convex, $F : K \multimap \mathbb{E}$ is continuous and condensing w.r.t. Hausdorff (or Kuratowski) measure of noncompactness. If F is inward, i.e, $F(x) \in x + T_K(x))$ for $x \in K$, then F has a fixed point.

Thus far we considered equations of the form

$$f(x) = 0 \text{ or } F(x) = x, \ \ x \in K,$$

i.e., constrained to some closed $K \subset \mathbb{E}$, assuming that $f, F : K \to \mathbb{E}$ is continuous compact and subject to additional conditions (tangency, inwardness).

Now we turn to problems of the form

$$0 \in Ax + F(x), \ \ x \in K,$$

where $F : K \to \mathbb{E}$ is continuous , $A : D(A) \to \mathbb{E}$ is a densely defined ω-dissipative operator for some $\omega \in \mathbb{R}$, i.e.: $\|\lambda x - Ax\| \geqslant (\lambda - \omega)\|x\|$ for $x \in D(A)$, $\lambda > \omega$ and $R(\lambda_0 I - A) = \mathbb{E}$ for some $\lambda_0 > \omega$ ($\Leftrightarrow \varrho(A) \cap (\omega, \infty) \neq \emptyset$) .

▲ *Equivalently (for linear) A is the generator of a C_0-semigroup $\{S_A(t)\}_{t \geqslant 0}$ of bounded linear operators on $\mathbb{E}$, or • $A = \partial\varphi$, where $\mathbb{E}$ is Hilbert and $\varphi : \mathbb{E} \to \mathbb{R}$ is convex lower semicontinuous.*

▲ *Constrained elliptic BVP*

$$(CEP) \qquad -\Delta u = f(x,u,\nabla u), \;\; x\in\Omega, \;\; u(x)\in\mathcal{K}; \;\; u|_{\partial\Omega}=0 \; \left(or \; \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}=0\right)$$

and its (strong) solutions.

▲ *Constrained parabolic initial BVP*

$$(CPP) \quad \dot u(t)-\Delta u = f(t,x,u,\nabla u), \;\; t\in[0,T], \; u(0,\cdot)=u_0, \; u|_{\partial\Omega}=0, \; u(t,x)\in\mathcal{K}$$

and its mild (or strong) solutions and periodic trajectories.

Here:

- $f:\Omega\times\mathbb{R}^N\times\mathbb{R}^{NM}\to\mathbb{R}^N$ (or $\varphi:[0,T]\times\Omega\times\mathbb{R}^N\times\mathbb{R}^{NM}\to\mathbb{R}^N$) is continuous;
- $\Omega\subset\mathbb{R}^M$ is a bounded domain in $\mathbb{R}^M$ with smooth boundary $\partial\Omega$;
- $\mathcal{K}\subset\mathbb{R}^N$ is closed (convex) set of state constraints.

Setting

- For $u = (u_1, ..., u_N) \in D(A) := H_0^1(\Omega) \cap H^2(\Omega, \mathbb{R}^N)$

$$Au = (\Delta u_1, ..., \Delta u_N),$$

where Δ is the usual Laplacian with Neumann (or Dirichlet) boundary conditions.

- $\mathbb{E}_0 = H^1(\Omega, \mathbb{R}^N)$, $\mathbb{E} = L^2(\Omega, \mathbb{R}^N)$,
  $K_0 := \{u \in \mathbb{E} \mid u(x) \in \mathcal{K} \text{ a.e. on } \Omega\}$, $K := \{u \in \mathbb{E} \mid u(x) \in \mathcal{K} \text{ a.e. on } \Omega\}$;

- (for (CEP)) $F(u) := f(\cdot, u(\cdot), \nabla u(\cdot))$, $u \in E_0$;
- (for (CPP)) $F(t, u) := f(t, \cdot, u(\cdot), \nabla(\cdot))$, $t \in [0, T]$, $u \in \mathbb{E}_0$.

Abstract problem

$$0 = Au + F(u), \ \ u \in K \cap D(A),$$

$$u' = Au + F(t, u), \ \ u(0) = u_0, \ u(t) \in K \cap D(A) \text{ for } t \in [0, T].$$

where

▲ $A : D(A) \to \mathbb{E}$ is a densely defined (in $\mathbb{E}$) linear operator,

$$D(A) \subset \mathbb{E}_0 \overset{j}{\hookrightarrow} \mathbb{E};$$

▲ $K_0 \subset \mathbb{E}_0$, $K \subset \mathbb{E}$ are closed sets, $j(K_0) \subset K$;

▲ $F : K_0 \to \mathbb{E}$ is continuous.

- For any $h > 0$, $h^{-1} \in \varrho(A)$, the resolvent

$$J_h := (I - hA)^{-1}$$

is well-defined.

Theorem (AĆ, WK (JDE 2011))

Assume that a closed $K \subset \mathbb{E}$ is a bounded $\mathcal{L}$-retract such that:
(1) The resolvent invariance, i.e. $J_h(K) \subset K$ for all $h > 0$;
(2) $F : K \to \mathbb{E}$ is continuous, tangent to K, i.e., $F(u) \in T_K(u)$, $u \in K$;
(3) A is ω-dissipative, $\varrho(A) \cap (\omega, \infty) \neq \emptyset$ and $D(A) \hookrightarrow\hookrightarrow \mathbb{E}$.
Then:
(i) the Euler characteristic $\chi(K)$ is well-defined;
(ii) if $\chi(K) \neq 0$ and K is bounded, then there is a solution to (CEP). i.e., there is $\bar{u} \in K \cap D(A)$ such that $0 \in Au + F(u)$.

Remark

(a) If (1), then K is semigroup invariant, i.e., $S_A(t)K \subset K$; if K is convex ten these conditions are equivalent.
(b) If $\mathcal{K}$ is convex, then it holds when A has $(\omega, +\infty) \subset \varrho(A)$ and $D(A) \hookrightarrow\hookrightarrow \mathbb{E}$.

Theorem (J. Siemianowski, WK (JFA to appear))

Suppose $K_0 \subset \mathbb{E}_0$ and $K \subset \mathbb{E}$ are closed convex bounded, $j(K_0) \subset K$ and:
- *A is densely defined, $(\omega, +\infty) \subset \varrho(A)$ for some $\omega \in \mathbb{R}$, $D(A) \hookrightarrow\hookrightarrow \mathbb{E}_0$;*
- *$J_h(K) \subset K_0$ for $h > 0$ with $h\omega < 1$;*
- *$F : K_0 \to \mathbb{E}$ is continuous, bounded and $F(u) \in T_K(j(u))$ for $u \in K_0$,*

then there is $\bar{u} \in D(A) \cap K_0$ with $0 = A\bar{u} + F(\bar{u})$.

In the proof: one takes $r : \mathbb{E} \to K$ (exists) such that

$$\|r(x) - x\| \leqslant 2d(x; K).$$

For small $h > 0$, with $h\omega < 1$, the fixed point problem

$$u = J_h(r(j(u) + hF(u)), \quad u \in K,$$

has a solution $u_h \in D(A) \cap K$, i.e., $\xi_h := u_h - hAu_h = r(j(u_h) + hF(u_h)) = r(w_h)$. One shows that

$$Au_h + F(u_h) \in T_K(j(u_h)), \quad w_h - r(w_h) = w_h - \xi_h = h(Au_h + F(u_h)).$$

The left expression is „perpendicular" to K, the right one is tangent: hence both are almost zero; passing $h \to 0$ yields the required $\bar{u} \in K \cap D(A)$.

• Let $f : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy Carathéodory conditions and sublinear growth.

• Let $K \subset \mathbb{R}^N$ be closed convex and bounded subset. We assume that $-f$ is tangent to K with respect to the second variable, i.e.

$$-f(x, u, v) \in T_K(u) \quad \text{for every } u \in K,\ x \in [0, T],\ v \in \mathbb{R}^N.$$

Theorem (JS, WK (2015))

The following Neumann bvp:

$$\begin{cases} \ddot{u}(x) = f(x, u(x), \dot{u}(x)) \text{ a.e. on } [0, T], \\ \dot{u}(0) = \dot{u}(R) = 0. \end{cases} \tag{1}$$

has a solution in $u \in W^{2,1}([0, T], \mathbb{R}^N)$ *such that* $u(x) \in K$ *for all* $x \in [0, T]$.

Let $\mathcal{K} \subset \mathbb{R}^N$ is closed convex and bounded.

- Let $f : \Omega \times \mathcal{K} \times (\mathbb{R}^M)^N \to \mathbb{R}^N$ such that:

- $f(\cdot, u, v) : \Omega \to \mathbb{R}^N$ is measurable; $f(x, \cdot, \cdot) : \mathcal{K} \times (\mathbb{R}^M)^N \to \mathbb{R}^N$ is continuous;
- There is $b \in L^2(\Omega)$ such that $\|f(x, u, v)\| \leqslant b(x)$ for a. e. $x \in \Omega$ and all $u \in \mathcal{K},\ v \in (\mathbb{R}^M)^N$.
- $f(x, u, v) \in T_{\mathcal{K}}(u)$ for a.a. $x \in \Omega$ and all $u \in K, v \in (\mathbb{R}^M)N$.

- Consider the system of N of nonlinear Poisson equations with Neumann BVP:

$$\begin{cases} -\Delta u(x) = f(x, u(x), \nabla u(x)) & \text{in } \Omega, \\ \dfrac{\partial u_1(x)}{\partial n} = \ldots = \dfrac{\partial u_N(x)}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\nabla u = (\nabla u_1, \ldots, \nabla u_N)$, $\dfrac{\partial u_i}{\partial n}$ denotes the outward normal derivative of u_i.

Theorem (JS, WK (2015))

There is a solution $u \in H^2(\Omega, \mathbb{R}^N)$ such that $\dfrac{\partial u}{\partial n} = 0$ in the sense of trace.

Recall that $K_0 = \{u \in \mathbb{E}_0 = H^1(\Omega, \mathbb{R}^N) \mid u(x) \in \mathcal{K}\}$,
$K = \{u \in \mathbb{E} = L^2(\omega, \mathbb{R}^N) \mid u(x) \in \mathcal{K}\}$. $A = \Delta$ with Neumann boundary condition.

In particular one has to show that for each $h > 0$,

$$(I - h\Delta)^{-1}(K) \subset K_0,$$

i.e. sort of a maximum principle:

Theorem (JS, WK (2015))

If $f \in K$, $h > 0$ and $u \in D(\Delta)$ such that

$$u - h\Delta u = f,$$

then $u \in K_0$.

Theorem (Bishop-Phelps (1978))

If $K \subset \mathbb{E}$, $\mathbb{E}$ separable, is closed convex, then it is the intersection of a countable collection of closed halfspaces that support it.

The consequence of the modified separation theorem.

The degree for elliptic constrained problems

The topological degree (the homotopy invariant) responsible for the existence of zeros of:

(1) (Miranda situation) $\varphi : K \to \mathbb{E}$, where $K \subset \mathbb{E}$ is a locally compact $\mathcal{L}$-retract, φ is weakly tangent to K, usc with closed convex values (or strongly tangent to K and admissible) ;
(2) (Deimling situation) $I - \varphi : K \multimap \mathbb{E}$, where $K \subset \mathbb{E}$ is closed convex, φ is weakly inward to K usc, condensing with convex compact values;
(3) (Elliptic situation) $A + F : K \to \mathbb{E}$, where K is an $\mathcal{L}$-retract, A generates a compact semigroup, F is weakly tangent to K, H-usc with weakly compact convex values.

• The Conley index approach for (CEP) with $\Omega = \mathbb{R}^N$: no compactness of the semigroup generated by the Schrödinger operator $-\Delta + V$.

Constrained parabolic problems

We consider the problem of the form

$$\dot{u} = Au + F(t, u)$$

and look for a periodic mild solution u, i.e. a continuous $u : [0, T] \to \mathbb{E}$ such that $u(0) = u(T)$

$$(CPP) \qquad u(t) = S_A(t)u(0) + \int_0^t S_A(t-s)w(s)\,ds,$$

where $w(s) \in F(s, u(s))$ for a.a. $s \in [0, T]$.

Theorem (JS, WK (2015))

(1) If K is closed convex and bounded, then (CPP) has a periodic solution.

(2) Suppose K is closed convex, $0 \in K$, F is bounded and spectral bound

$$s(A) := \{\mathrm{Re}\,\lambda \mid \lambda \in \sigma(A)\} < 0$$

(as for Δ), then (CPP) has a periodic solution.

Hopf bifurcation..., periodic solutions without boundedness..., multiplicity of solutions...

References

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THANK YOU „Tangency is everywhere"

This would be all (the survey paper of Mawhin (2013); tens of new proofs, rediscoveries, applications and a recent preprint Fonda-Gidoni (2015)) if not for.... WHAT FOLLOWS.

Robin boundary condition

- Let $u \in H^1(\Omega)$. We say that $\Delta u \in L^2(\Omega)$ is there is $f \in L^2(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = -\int_\Omega fv$$

for all $v \in C_0^\infty(\Omega)$. In this case we let $\Delta u := f$ (correct definition).

- Let $u \in H^1(\Omega)$ and $\Delta \in L^2(\Omega)$. We say that $\frac{\partial u}{\partial \nu} \in L^2(\Gamma)$ if there is $b \in L^2(\Gamma)$ such that

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega \Delta uv = \int_\Gamma bv|_\Gamma$$

for all $v \in H^1(\Omega)$. In this case we let $\frac{\partial u}{\partial \nu} := b$ (correct definition).

These definitions are such that the Green Formula

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega \Delta uv = \int_\Gamma \frac{\partial u}{\partial \nu} v|_\Gamma$$

(which is „clasically" valid for $u, v \in C^2(\overline{\Omega})$) holds for all $v \in H^1(\Omega)$, whenever $u \in H^1(\Omega)$, $\Delta u \in L^2(\Omega)$ and $\frac{\partial u}{\partial \nu} \in L^2(\Omega)$.

- Let $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ be a Hilbert space, $(V, \|\cdot\|)$ a Banach space such that $j : V \hookrightarrow \mathbb{H}$ ($V \subset \mathbb{H}$ and $j : V \to \mathbb{H}$, $j(v) := v$ is continuous) and V is dense in $\mathbb{H}$.

Fact (Gelfand evolution triple)

After the identification $\mathbb{H} = \mathbb{H}^*$ (via the Riesz theorem), $i : \mathbb{H} \hookrightarrow V^*$ ($\langle v, i(h)\rangle := \langle v, h\rangle_{\mathbb{H}}$, $h \in \mathbb{H}$, $v \in V$); if V is reflexive, then $\mathbb{H}^*$ is dense in V^*, i.e., $V \hookrightarrow \mathbb{H} \hookrightarrow V^*$.

- [Dirichlet form] Let a bilinear form $a : V \times V \to \mathbb{R}$ be symmetric, continuous and elliptic, i.e.,

$$a(v,v) + \omega\|v\|^2_{\mathbb{H}} \geqslant \alpha\|v\|^2_V,$$

for all $v \in V$ and some $\omega \in \mathbb{R}$ and $\alpha > 0$.

- Define $\mathcal{A} : V \to V^*$ by

$$\mathcal{A}v(w) := a(v,w), \quad v, w \in V.$$

- Define $A : D(A) \to \mathbb{H}$, $D(A) \subset \mathbb{H}$, by

$$x \in D(A) \text{ and } y = Ax \iff x \in V \text{ and } a(x,v) = \langle y, v\rangle_{\mathbb{H}} \ \forall\, v \in V.$$

In other words $A = \mathcal{A}|_{D(A)}$. We say that $\mathcal{A}$, A are generated by the form a.

Theorem

Let $V \hookrightarrow \mathbb{H} \hookrightarrow V^*$ be a Gelfand triple, $a : V \times V \to \mathbb{R}$ a Dirichlet form (i.e., symmetric, continuous and elliptic). If $\mathcal{A} : V \to V^*$ and $A : D(A) \to \mathbb{H}$ are generated by a, then:

(1) $\mathcal{A}$ is symmetric (i.e., $\langle w, \mathcal{A}v\rangle = \langle v, \mathcal{A}w\rangle$), linear and bounded.

(2) $D(A) \subset V$ is a dense (linear) subspace in $\mathbb{H}$, A is self adjoint linear and closed and, in general unbounded, but $\langle Au, u\rangle_{\mathbb{H}} \geqslant -\omega\|u\|^2_{\mathbb{H}}$;

(3) $-A$ is the generator of an analytic C_0 semigroup $\{S_A(t); \mathbb{H} \circlearrowleft\}_{t\geqslant 0}$ such that

$$\|S_A(t)\|_{\mathbb{H}} \leqslant e^{\omega t}, \quad t \geqslant 0.$$

(4) The resolvent set $\varrho(A) \supset (\omega, +\infty)$, $\|(\lambda I + A)^{-1}\|_{\mathcal{L}(\mathbb{H})} < \frac{1}{\lambda - \omega}$ for $\lambda > \omega$.

(5) If $\omega = 0$, i.e., a is coercive, then $\varrho(-A) \supset (-\gamma, +\infty)$ for some $\gamma > 0$; in particular $A^{-1} : \mathbb{H} \to D(A)$ is well-defined and continuous with $\|A^{-1}\| \leqslant \frac{1}{\gamma}$.

(6) A has compact resolvents $\iff$ the embedding $V \hookrightarrow \mathbb{H}$ is compact; hence the semigroup $\{S_A(t)\}$ is compact, i.e., $S_A(t)$ is compact for all $t > 0$. The spectrum $\sigma(A)$ consists then of eigenvalues only.

(7) If $V \hookrightarrow \mathbb{H}$ is compact and $\mathbb{H}$ is separable, then $\mathbb{H}$ basis $(e_n)_{n=1}^{\infty}$ such that

$$Ae_n = \lambda_n e_n, \quad \text{where for } n \in \mathbb{N}\ \lambda_n \leqslant \lambda_{n+1}, \quad \lim_{n\to\infty} \lambda_n = +\infty.$$

One has $\lambda_n = \sup\{\min_{u\in W_1} a(u,u) \mid W \subset V,\ \dim V/W = n-1\}$ here $W \subset V$ is a closed subspace and $W_1 := \{u \in W \mid \|u\|_{\mathbb{H}} = 1\}$.