Lodz University of Technology

Invariance and Strict Invariance of Evolution Equations

Wojciech Kryszewski

Analysis in Tatra
Seminar for Researchers

Małe Ciche, September 7 - 11, 2022

Equation

We consider the following problem

Equation

$$\begin{cases} \dot{u} \in -Au + f(t,u), \; t \geqslant 0, \\ u(0) = x_0, \end{cases}$$

where:

- $A : D(A) \to X$ is a quasi m-accretive operator in a Banach space X;
- $f : [0,+\infty) \times \Omega \to X$, $\Omega \subset X$ is open, a continuous perturbation;
- $x_0 \in \Omega \cap \overline{D(A)}$ is the initial state.

Meaning

- An (appropriately defined) solution $u:[0,T)\to X$ correspond to the state of a system evolving, whose behavior is governed by A and f.
- Accretivity of A means that the energy of the spontaneous evolution of the system dissipates. By default A represents anisotropic diffusion combined with drift (advection) or damping.
- The perturbation f (depending on time and the state) represents the forcing term (which can be controlled).

Assume that a closed set $K\subset X$, $K\cap\Omega\neq\emptyset$, is distinguished and treated as the set of admissible state (or local) constraints for the state u. This means that the motion off K is possible but not welcome.

In other words the forcing term should prevent the system starting at some point $x_0\in K$ from escaping elsewhere.

Problems

Problems

1. (Existence, viability The existence of solutions $u:[0,T]\to X$ such that $u(0)=x_0$ and u survives in K, i.e. $u(t)\in K$ for all life-time $t\in[0,T)$;
2. (Structure of viable solutions) The study of the structure of the set of solutions surviving in K (in the spirit of the Aronszajn or Kneser theorem);
3. (Invariance) Any solution starting in K stays there forever;
4. (Strict invariance) Any solution starting in K later on stays in the interior of K.

The first issue is studied in many classic and recent papers. Issues 2, 3 and 4 make sense since there is no uniqueness of solutions. In the talk I will address the third and fourth issues, i.e.

We look for conditions implying the invariance of K, i.e. the system is invariant and strict invariant with respect to K.

Dissipation

Let H be a (real for simplicity) Hilbert space. A linear operator $A: D(A) \to H$ is said to be dissipative if either

$$\langle Au, u\rangle_H \leqslant 0, \;\; u \in D(A).$$

The operator A is maximally dissipative if it is not the proper restriction of any other dissipative operator.

The physical interpretation of this concept differs in detail depending on the physical context. The simplest way to describe dissipation in physical systems is by adding a resisting (damping) force to the Newtonian equation of motion. In principle one can derive the damped motion (or evolution) of the system by coupling the system to a heat bath or to the motion of other particles of a force field.

All in all, however, when dealing with an evolutionary system (e.g. hyperbolic system) treated via semigroup theory, we may rewrite the system as an abstract Cauchy problem

$$\begin{cases} \dot{u} = Au \\ u(0) = u_0, \end{cases}$$

where A is an operator on a suitable Hilbert H space chosen so that the energy of the system (or, precisely, the expression proportional to the (kinetic) energy),

$$E(t) := \|u(t)\|_H^2 = \langle u(t), u(t)\rangle_H.$$

As a consequence

$$\dot{E}(t) = \langle \dot{u}(t), u(t)\rangle_H = \langle Au(t), u(t)\rangle_H \leqslant 0.$$

This means that A is dissipative if there is the dissipation of energy, i.e., the energy of the system is nonincreasing in time.

We define dissipative forces in classical dynamics as any and all types of interaction where the energy is lost when the evolution takes place (usually in the form of heat to a heat bath [i.e. a system whose heat capacity is so large that when it is in thermal contact with some other system of interest its temperature remains constant. The heat bath is effectively an infinite reservoir of energy and accessible quantum states at a given temperature]

In case $A : D(A) \to H$ is not linear, then one speaks of monotonicity. An operator (possibly multivalued) is monotone if

$$\langle x-y, u-v\rangle_H \geqslant 0$$

for all $x, y \in D(A)$ and $u \in Ax$, $v \in Ay$.

When solving

$$Au = f$$

one first solves a sequence of simpler, approximate problems

$$A_\varepsilon(u_\varepsilon) = f,\ \varepsilon > 0,$$

Having u_ε, if A_ε is monotone, then fixing $\varphi \in D(A)$

$$0 \leqslant \langle u_\varepsilon - \varphi, A_\varepsilon(u_\varepsilon) - A_\varepsilon(\varphi)\rangle_H = \langle u_\varepsilon - \varphi, f - A_\varepsilon(\varphi)\rangle_H.$$

Suppose $A_\varepsilon(\varphi) \to A(\varphi)$, $u_\varepsilon \to u$ as $\varepsilon \searrow 0$; then

$$0 \leqslant \langle u - \varphi, f - A(\varphi)\rangle_H.$$

Then, provided $D(A)$ is dense in H, the Minty maximal monotonicity trick implies that

$$A(u) \ni f.$$

Trying to „lift" the Minty trick to Banach spaces leads to the notion of a nonlinear accretive or dissipative operators

Accretive and dissipative operators (Brezis-Nirenberg)

Let X ba (real) Banach space.

- A (possible multivalued) operator $A : D(A) \to X$ is dissipative if

$$[x-y, u-v]_- \leqslant 0 \quad \forall (x,u),(y,v) \in \mathrm{Gr(A)} := \{(\mathrm{x},\mathrm{u}) \mid \mathrm{u} \in \mathrm{Ax}\}.$$

- A is accretive if

$$[x-y, u-v]_+ \geqslant 0 \quad \forall (x,u),(y,v) \in \mathrm{Gr(A)}.$$

Here the right semi-inner product

$$[x,y]_\pm := \lim_{h \to 0^\pm} \frac{\|x+hy\| - \|x\|}{h},$$

i.e. $[x,y]_\pm$ is the lower right (resp. left) Dini directional derivative of $\|\cdot\|$ at x in the direction of y.

There is another, perhaps more intuitive description of the semi-inner product

$$\|x\|\cdot[x,y]_\pm=\langle x,y\rangle_\pm,$$

where

$$\langle x,y\rangle_+:=\sup_{p\in J(x)}p(y),\quad \langle x,y\rangle_-:=\inf_{p\in J(x)}p(y)$$

and

$$J(x):=\{p\in X^*\mid p(x)=\|x\|^2=\|p\|^2\}$$

is the (normalized) duality mapping in X.

• $\langle\cdot,\cdot\rangle_\pm$ has most of natural properties of the inner product $\langle\cdot,\cdot\rangle_H$ in a Hilbert space H; actually if $X=H$ is a Hilbert space, then

$$\langle\cdot,\cdot\rangle_H=\langle\cdot,\cdot\rangle_\pm.$$

• In particular If $u:[a,b]\to X$ is right (left) differentiable at $t\in[a,b]$, then so is $\|u(t)\|$ and

$$\frac{d^\pm}{dt}\|u(t)\|=\left[u(t),\frac{d^\pm}{dt}u(t)\right]_\pm.$$

- The operator A is *m-accretive* if it is accretive and

$$R(I+\lambda A) := \{y \in X \mid y \in x + \lambda Ax \text{ forsome } x \in D(A)\} = X.$$

- A is quasi accretive or quasi *m*-accretive if there is $\alpha \in \mathbb{R}$ such that $\alpha I + A$ is accretive (resp. m-accretive), i.e.

$$[x-y, u-v]_+ \geqslant -\alpha\|x-y\| \quad \forall (x,u),(y,v) \in \mathrm{Gr(A)}.$$

The terminology for dissipative operators (with $[\cdot,\cdot]_-$ instead) is analogous.

Fact

- The operator A is dissipative (resp. m-dissipative, quasi dissipative) iff $-A$ is accretive (resp. m-accretive, quasi accretive).
- If A is m-accretive, the it is maximal accretive.
- If $X = H$ is a Hilbert space, then A is accretive iff A is monotone; in this case m-accretivity coincides with maximal monotonicity.

Assume $A : D(A) \to X$ is a α-m-accretive operator; let $T > 0$ and $w \in L^1([0,T],X)$. Consider the problem

$$(*) \qquad \begin{cases} \dot{u}(t) \in -Au(t) + w(t), & t \in [0,T], \\ u(0) = x \in \overline{D(A)}. \end{cases}$$

Solutions

(a) A continuous $u : [0,T] \to X$, $T > 0$, is a strong solution if $u \in W^{1,1}_{loc}((0,T],X)$, $u(t) \in D(A)$, $u(0) = x$ and $\dot{u}(t) - w(t) \in Au(t)$ for a.a. $t \in (0,T]$.

- Strong solutions rarely exist. For instance if X is reflexive, $x \in D(A)$ and $w \in W^{1,1}([0,T],X)$, then there is a unique strong solution $u \in W^{1,\infty}([0,T],X)$.

(b) A continuous $u : [0,T] \to X$ is an integral solution if $u(0) = x$ and for any $0 \leqslant s \leqslant t \leqslant T$ and $(y,v) \in \mathrm{Gr}(A)$,

$$e^{-t\alpha}\|u(t) - y\| \leqslant e^{-s\alpha}\|u(s) - y\| + \int_s^t e^{-z\alpha}\left[u(z) - y, w(z) - v\right]_+ dz.$$

- It is known that a strong solution is integral. The problem always admits a unique integral solution denoted by $u = u_A(\cdot; x, w) : [0,T] \to X$ and $u(t) \in \overline{D(A)}$ for all $t \in [0,T]$.

Consider the problem

$$(**)\qquad \begin{cases} \dot u(t) \in -Au(t) + f(t, u(t)), & t \in [0, T], \\ u(0) = x \in \overline{D(A)}, \end{cases}$$

where $f : [0, +\infty) \times \Omega \to X$ is a Carathéodory map (satisfying some appropriate growth conditions).

Given $T > 0$, a continuous $u : [0, T] \to \Omega$ is an integral solution if it solves $(*)$ with $w := f(\cdot, u(\cdot))$. A function $u : [0, \tau) \to \Omega$, $0 <\leqslant \tau\infty$, is an integral solution if for any $0 < T < \tau$, $u|_{[0,T]}$ is a solution in the above sense.

- Originally (Brezis) solution to problems of the form $(*)$ or $(**)$ have been considered in the approximate sense (as limits of some approximating sequence of the so-called ε-DS-solutions). The distinction of the notion of an integral solution corresponding to the usual concept of a solution is due to the late Ph. Benilan, J. Crandall 2001 and V. Barbu.

Adjustments

- Let A be α-m-accretive. Then $B := A + \alpha I$ is m-accretive. Hence instead of

$$(**) \qquad \begin{cases} \dot{u}(t) \in -Au(t) + f(t, u(t)), & t \in [0, T], \\ u(0) = x \in \overline{D(A)}, \end{cases}$$

we can consider

$$(***) \qquad \begin{cases} \dot{u}(t) \in -Bu(t) + g(t, u(t)), & t \in [0, T], \\ u(0) = x \in \overline{D(A)}, \end{cases}$$

where $g(t,u) := f(t,u) + \alpha u$, $u \in \Omega$, $t \geqslant 0$. This means that without loss of generality one may study m-accretive operators only since (integral) solutions to $(**)$ and $(***)$ coincide.

- Given an m-accretive $A : D(A) \to X$ let

$$C := A(\cdot) \times 0 : D(A) \times \mathbb{R} \to X \times \mathbb{R}, \; C(u,t) = (Au, 0), \; u \in D(A), \; t \in \mathbb{R}.$$

and $F(U) = (f(t,u), 1)$, $U = (u,t) \in \Omega \times \mathbb{R}$. Then C is m-accretive. Solutions to $(**)$ coincide with solution $U = (u(\cdot), \cdot)$ to

$$\begin{cases} \dot{U} \in -CU + F(U), & t \in [0, T], \\ U(0) = (x, 0) \in \overline{D(B)}. \end{cases}$$

As the constraining set one takes $K \times [0, +\infty)$.

- In many applications the constraint set is of the form

$$K = \{x \in X \mid V(x) \leqslant 0\},$$

where $V : X \to \mathbb{R}$ is a locally Lipschitz potential.

This representation is not restrictive, since for any closed K we have

$$K = \{x \in X \mid d_K(x) \leqslant 0\},$$

where $d_K(x) := \inf_{k \in K} \|x - k\|$ is the distance function.

In what follows we study sufficient (and necessary) conditions for (strict) invariance of the closed set K stated in terms of the potential V for the equation

$$\begin{cases} \dot{u} \in -Au + f(u), \ t \geqslant 0, \\ u(0) = x, \end{cases}$$

where A is densely defined, i.e. $\overline{D(A)} = X$ m-accretive operator and $f : \Omega \to X$ is continuous.

Strategies

For simplicity let consider $A \equiv 0$ (in this case integral and C^1-solutions coincide)
There are, roughly speaking two strategies considered:

• One sided Lipschitz estimates on f, local viablity or the so-called tangency conditions (Romanian attitude: Barbu, Vrabie, Carja and others over 2005-2020). This conditions make it possible to compare any solution with those surviving in K and show that theu must remain in K ;

• Exterior tangency conditions of the form

$$D_+ d_K(x; f(x)) \leqslant \beta d_K(x)),\ x \in U,$$

where $D_+ d_K(x; v)$ denotes the directional lower right Dini derivative of the distance function d_K at x in direction of $v \in X$, $\beta > 0$, and $U \subset \Omega$ is an open neighborhood of $K \cap \Omega$ (French, Italian and German attitude - Bothe, Volkmann (1998- 2007, Cannarsa, da Prato, Frankowska (2019 - 2021)). The approach resembles that Lagrange stability arguments and in fact shows that if u is a solution, then the function $d_K(u(\cdot))$ is nonincreasing, i.e. if $u(0) \in K$, then $dK(u(0)) = 0$, so also $d_K(u(t)) \leqslant 0$ meaning that $u(t) \in K$ for all t.

The presence of A makes thing much more complicated:

- The first attitude requires additional strong assumptions concerning A and the good behavior of f with respect to the so-called A-modified tangent cone to K.
- The results of Cannarsa, Da Prato, Frankowska 2020, 2021 concern the case when $-A$ is the generator of a strongly continuous contractive semigroup on a reflexive space X and $f: X \to X$ is quasi dissipative. Their condition for invariance reads

$$D_+ d_K(x; -Ax + B(x)) \leqslant \beta d_K(x), \; x \in D(A) \cap U,$$

where U is a neighborhood of K.

Paradigms of both attitudes is similar (or much more complicated) as above.

In our study we follow the second approach rather.

A-derivative of functionals

Let $A : D(A) \to X$ be m-accretive, let $V : X \to \mathbb{R}$ be a locally Lipschitz functional (representing K), let $x \in X$ and let $v \in X$.

- By $u = u_A(\cdot; x, v) : [0, +\infty) \to X$ we denote the unique integral solution to

$$(*) \qquad \begin{cases} \dot{u}(t) \in -Au(t) + w(t), & t \in [0, T], \\ u(0) = x \in X, \end{cases}$$

where $w(t) = v$, $t \geqslant 0$, i.e. w is constant.

A-derivative

By the A-derivative of V at x in the direction of v we mean the lower right Dini derivative

$$D_A V(x; v) := \liminf_{h \to 0^+} \frac{V(u_A(h; x, v)) - V(x)}{h} = D_+(V \circ u_A(\cdot; x, v))(0).$$

This derivative measures the rate of growth of V along the integral curve $u_A(\cdot; x, v)$ in the neighborhood of x.

It appears that this is a very convenient concept when studying invariance issues

Theorem 1

If for every $z \in \partial K \cap \Omega$ there is a neighborhood $U(z) \subset \Omega$ of z and a uniqueness function ω such that

$$D_A V(x; f(x)) \leqslant \omega(V(x)) \text{ for } x \in U(z) \setminus K, \tag{1}$$

then K is invariant.

There are situations (in many applications), when the verification of condition (1) is not obvious, if even possible.

Theorem 2

Assume that X is reflexive and:

- for any $z \in \partial K \cap \Omega$, there is a neighborhood $U(z) \subset \Omega$ of z and $\beta_z > 0$ such that

$$[x-y, f(x)-f(y)]_+ \leqslant \beta_z \|x-y\| \text{ for } x, y \in U(z); \qquad (1)$$

$$D_A V(x; f(x)) \leqslant \omega(V(x)) \text{ for } x \in [U(z) \setminus K] \cap D(A), \qquad (2)$$

- f is bounded on bounded sets,

then K is invariant.

Comments

- the one sided Lipschitz estimate (1) implies the local uniqueness of solutions starting at point of the boundary. It may be relaxed but the condition is a bit more complicated.
- Reflexivity of X is sometimes too restrictive: in many applications one deals with L^1-space or the space of continuous functions.

Without reflexivity

Theorem 3

Suppose conditions of Theorem 2 are satisfied save condition (2). Instead we assume that

- for every $z \in \partial K \cap \Omega$ there are a neighborhood $U(z) \subset \Omega$ of z and a nondecreasing uniqueness function ω such that for every solution $u : [0, \tau) \to X$ of the problem surviving in $U(z)$ with $u(0) \in D(A)$ one has

$$D_+(V \circ u)(t) \leq \omega(V(u(t))) \text{ for } t \in (0, \tau) \text{ with } u(t) \in U(z) \setminus K,$$

then K is invariant.

The above condition differs from (2) in Theorem 2 since one has to control the rate of growth of V along all solutions, while in (2) this control is required only along $u_A(\cdot; x, f(x))$. This is the price one has to pay to relax the reflexivity property.

THANKS FOR YOUR ATTENTION