A Neohookean model of plates

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References

A Neohookean model of plates joint work with

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Introduction

We study hyperelastic deformations of neohookean materials in planar domains, called plates. These problems are motivated by recent remarkable relations between Geometric Function Theory (GFT) and the theory of Nonlinear Elasticity (NE).
Both theories are governed by variational principles. We confine ourselves to deformations of bounded Lipschitz planar domains $\mathbb{X}$ and $\mathbb{Y}$.

The general theory of hyperelasticity deals with:

- Sobolev homeomorphisms $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$
- $J_h \stackrel{\text{def}}{=\!=} \det Dh \geqslant 0$;
- which minimize the stored energy functional

$$\mathcal{E}[h] = \int_{\mathbb{X}} E\left(|Dh|, \det Dh\right) \, \mathrm{d}x, \quad \text{where } E \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}.$$

The stored energy is determined by the elastic and mechanics properties of the material.
Hereafter $Dh \in \mathbb{R}^{2\times 2}$ stands for the deformation gradient and $|Dh|$ its Hilbert-Schmidt norm.

$\mathscr{H}^p(\mathbb{X}, \mathbb{Y})$ space

We are concerned with orientation-preserving homeomorphisms $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$, denoted by $\mathscr{H}^p(\mathbb{X}, \mathbb{Y})$. The term neohookean refers to a stored energy function E which increases to infinity when J_h approaches zero. A model example is

$$\mathsf{E}_q^p[h] = \int_{\mathbb{X}} \left[|Dh|^p + \frac{1}{(\det Dh)^q} \right] \mathrm{d}x \, , \quad p > 1 \text{ and } q > 0 \, . \tag{1}$$

We assume that the class of admissible homeomorphisms is nonempty; that is, there is $h \in \mathscr{H}^p(\mathbb{X}, \mathbb{Y})$ such that $\mathsf{E}_q^p[h] < \infty$.

We shall accept the weak limits of energy-minimizing sequences of homeomorphisms as legitimate deformations.

We allow the so called weak interpenetration of matter, squeezing of the material can occur.

Remark. This changes the nature of minimization problem to the extent that minimal energy (usually attained) can be strictly smaller than the infimum energy among homeomorphisms

How to enlarge the class of homeomorphisms as little as possible to ensure the existence of minimizers in that class?

monotone Sobolev mappings

Indeed, that such a class is a bare minimal enlargement of homeomorphisms follows from a Sobolev variant of Young's theorem.

A continuous map between compact oriented topological 2-manifolds (such as plates and thin films) is monotone if and only if it is a uniform limit of homeomorphisms.

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Definition (C.B. Morrey) Monotonicity simply means that for a continuous $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ the preimage $h^{-1}(C)$ of a continuum $C \subset \overline{\mathbb{Y}}$ is a continuum in $\overline{\mathbb{X}}$.

The Sobolev variant reads as

Theorem (*approximation by Sobolev homeomorphisms*)

Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded Lipschitz planar domains. Suppose that $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ is a monotone (continuous) Sobolev mapping in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^2)$, $1 < p < \infty$. Then h can be approximated strongly in norm topology $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^2)$ and uniformly by homeomorphisms $h_j : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$.

Let $\mathscr{M}^p(\overline{\mathbb{X}}, \overline{\mathbb{Y}})$ denote the class of orientation preserving monotone mappings $h \colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C})$.

Theorem (Main 1)

Let $p \geqslant 2$ and $q > 0$. Then there exists $h_\circ \in \mathscr{M}^p(\overline{\mathbb{X}}, \overline{\mathbb{Y}})$ such that

$$\mathsf{E}_p^q[h_\circ] = \inf_{h \in \mathscr{M}^p(\overline{\mathbb{X}}, \overline{\mathbb{Y}})} \mathsf{E}_q^p[h] \tag{2}$$

Theorem (Main 2)

Let $p > 2$ and $q \geqslant \frac{p}{p-2}$. Then there exists a homeomorphism $h_\circ \in \mathscr{H}^p(\mathbb{X}, \mathbb{Y})$ such that

$$\mathsf{E}_q^p[h_\circ] = \inf_{h \in \mathscr{M}^p(\overline{\mathbb{X}}, \overline{\mathbb{Y}})} \mathsf{E}_q^p[h] = \inf_{h \in \mathscr{H}^p(\mathbb{X}, \mathbb{Y})} \mathsf{E}_q^p[h] .$$

The existence of monotone minimizer $h_\circ$ is ensured by Theorem (Main 1) and the fact that $h_\circ$ is a homeomorphism follows by:

Theorem

Let $p > 2$ and $q \geqslant \frac{p}{p-2}$. If $h \in \mathscr{M}^p(\overline{\mathbb{X}}, \overline{\mathbb{Y}})$ and

$$\int_\Omega \left(|Dh|^p + \frac{1}{J_h^q} \right) \mathrm{d}x < \infty \quad \text{for every subdomain } \Omega \subset\subset \mathbb{X},$$

then $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a homeomorphism.

Remark 1. Theorem (Main 2) also holds for $p = 2$ and $q = \infty$, in which case the finite energy condition should be understood as:

$$\int_{\mathbb{X}} |Dh|^2 \, \mathrm{d}x < \infty \quad \text{and} \quad \frac{1}{\det Dh} \in \mathscr{L}^\infty(\mathbb{X}),$$

meaning that $\det Dh(x) \geqslant \frac{1}{C} > 0$ for a constant $C = \|J_h^{-1}\|_{\mathscr{L}^\infty(\mathbb{X})}$.
Remark 2. Theorem (Main 2) fails if $0 < q < \frac{p}{p-2}$

Example

For $0 < q < \frac{p}{p-2}$, there exists a noninjective $h \in \mathscr{M}^p(\overline{\mathbb{X}}, \overline{\mathbb{Y}})$ with $\mathsf{E}_q^p[h] < \infty$.

Partial injectivity

This example raises a question about partial injectivity. We have:

Theorem

Suppose that a monotone map $h : \overline{\mathbb{X}} \to \overline{\mathbb{Y}}$ *of the Sobolev class* $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{C})$ *has a positive Jacobian determinant a.e. Then*

- *h is globally invertible in the sense that: For every $h \in \mathcal{A}^p$ with $p > 2$ and*

$$\mathcal{A}^p \stackrel{\text{def}}{=\!=} \{h \in \mathscr{C}(\overline{\mathbb{X}}, \mathbb{C}) \cap \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{C}) \colon J(x,h) > 0 \text{ a.e.}, h = \varphi \text{ on } \partial\mathbb{X}\}$$

  the following set has full measure:

$$\mathbb{Y}_h \;=\; \{y \in h(\overline{\mathbb{X}}) \colon h^{-1}(y) \text{ is a single point } \} \subset \overline{\mathbb{Y}} \qquad (3)$$

- *There exists $\mathbb{X}_h$ of full measure in $\mathbb{X}$ such that h restricted to $\mathbb{X}_h$ is injective.*

Study of the *branch set*

Study of

$$\mathcal{B}_h \overset{\text{def}}{=\!=} \{x \in \mathbb{X} \colon h \text{ fails to be homeomorphic near } x\}$$

and its image $h(\mathcal{B}_h)$.

Example

If $0 < q < \frac{p}{p-2}$, then there exists $h \in \mathscr{M}^p(\overline{\mathbb{X}}, \overline{\mathbb{Y}})$ with $\mathsf{E}_q^p[h] < \infty$ such that $|\mathcal{B}_h| > 0$ and $|h(\mathcal{B}_h)| > 0$.

Our example is based on a Cantor type construction.

Important lemma

Lemma

Let $Q \subset \mathbb{R}^2$ *be an arbitrary square. For every* $p > 2$ *and* $0 < q < \frac{p}{p-2}$ *there exists a non-injective monotone map* $\Phi \in \mathscr{M}^p(\mathrm{Q}, \mathrm{Q})$ *of finite* E_q^p*-energy, which is the identity near the boundary of* Q*.*

Cornersquares. Suppose we are given a square $Q = I \times J \subset \mathbb{R}^2$ (I and J closed intervals of the same lenght) and a parameter $0 < \varepsilon < 1$.
The notation εI and εJ will stand for the intervals of the same centers but ε-times smaller in length, respectively.
Cutting them out from I and J gives the decompositions:

$$I \setminus \varepsilon I = I_- \cup I_+ \text{ and } J \setminus \varepsilon J = J^- \cup J^+$$

into the left and the right, as well as into the lower and the upper subintervals.

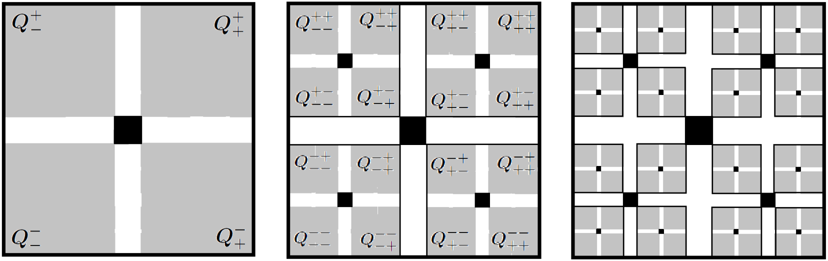
The Cartesian product consists of four subsquares:

$$(I \setminus \varepsilon I) \times (J \setminus \varepsilon J) \;=\; Q_+^+ \cup Q_-^+ \cup Q_-^- \cup Q_+^- \,.$$

Explicity

$$Q_+^+ \stackrel{\text{def}}{=\!=} I_+ \times J^+ \,,\; Q_-^+ \stackrel{\text{def}}{=\!=} I_- \times J^+ \,,\; Q_-^- \stackrel{\text{def}}{=\!=} I_- \times J^- \,,\; Q_+^- \stackrel{\text{def}}{=\!=} I_+ \times J^- \,.$$

Each of these sub-squares touches exactly one corner of Q, which motivates our calling Q_+^+, Q_-^+, Q_-^-, Q_+^- the *cornersquares* of Q; more precisely, the first generation of cornersquares.
We shall also spot the so-called *centersquare* of Q, defined by $\varepsilon Q = \varepsilon I \times \varepsilon J$.



$Q^{\beta_1}_{\alpha_1} \in \mathcal{F}_1(\mathbb{Q})$ $Q^{\beta_1\beta_2}_{\alpha_1\alpha_2} \in \mathcal{F}_2(\mathbb{Q})$ $Q^{\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3} \in \mathcal{F}_3(\mathbb{Q})$

Cornersquares of the first, second and third generation
The branch set will develop in the black centersquares

Figure: Cornersquares as building blocs for a Cantor type construction

Second generation of cornersquares

Choose another positive ε-parameter, say $\varepsilon = \varepsilon_2$. Then every cornersquare of Q gives rise to its own four cornersquares determined by this parameter, see the middle part of Figure 1. In this way we obtain sixteen cornersquares of so-called second generation. According to our notation these are:

$$\begin{matrix} Q^{++}_{++} & Q^{++}_{+-} & Q^{+-}_{+-} & Q^{+-}_{++} \\ Q^{++}_{-+} & Q^{++}_{--} & Q^{+-}_{--} & Q^{+-}_{-+} \\ Q^{-+}_{-+} & Q^{-+}_{--} & Q^{--}_{--} & Q^{--}_{-+} \\ Q^{-+}_{++} & Q^{-+}_{+-} & Q^{--}_{+-} & Q^{--}_{++} \end{matrix}$$

See also the third generation of 64 cornersquares in the right hand side of Figure 1.

The induction procedure

Fix a sequence of ε-parameters rapidly decreasing to 0, say $(\varepsilon_1, \varepsilon_2, ...)$ with $\varepsilon_n = 4^{-n}$. We begin with the base 1×1 square $\mathbb{Q} \subset \mathbb{R}^2$ and the first ε- parameter equal to ε_1. This gives us the first generation of four cornersquares $Q_{\alpha_1}^{\beta_1} \subset \mathbb{Q}$, where both indices run over the set $\{+\,,\,-\}$. We let $\mathcal{F}_1$ denote this family of cornersquares.
In the second step we take ε_2 as the ε-parameter and look at the cornersquares of every $Q_{\alpha_1}^{\beta_1}$. Denote them by $Q_{\alpha_1\,\alpha_2}^{\beta_1\,\beta_2} \subset Q_{\alpha_1}^{\beta_1}$, where $\alpha_2, \beta_2 \in \{+\,,\,-\}$. They form the family $\mathcal{F}_2$ of second generation.

More generally, given the family $\mathcal{F}_n$ of cornersquares $Q^{\beta_1\,\beta_2\ldots\beta_n}_{\alpha_1\,\alpha_2\ldots\alpha_n} \subset Q^{\beta_1\,\beta_2\ldots\beta_{n-1}}_{\alpha_1\,\alpha_2\ldots\alpha_{n-1}} \in \mathcal{F}_{n-1}$, we take ε_{n+1} as the ε-parameter and adopt to the family $\mathcal{F}_{n+1}$ the ε-cornersquares of $Q^{\beta_1\,\beta_2\ldots\beta_n}_{\alpha_1\,\alpha_2\ldots\alpha_n}$; namely,

$$Q^{\beta_1\,\beta_2\ldots\beta_n+}_{\alpha_1\,\alpha_2\ldots\alpha_n+}\,,\; Q^{\beta_1\,\beta_2\ldots\beta_n+}_{\alpha_1\,\alpha_2\ldots\alpha_n-}\,,\; Q^{\beta_1\,\beta_2\ldots\beta_n-}_{\alpha_1\,\alpha_2\ldots\alpha_n-}\,,\; Q^{\beta_1\,\beta_2\ldots\beta_n-}_{\alpha_1\,\alpha_2\ldots\alpha_n+} \in \mathcal{F}_{n+1}\,.$$

Thus $\mathcal{F}_{n+1}$ consists of 4^{n+1} cornersquares denoted by $Q^{\beta_1\,\beta_2\ldots\beta_n\beta n+1}_{\alpha_1\,\alpha_2\ldots\alpha_n\alpha_{n+1}}$. This process continuous indefinitely.

The size of squares in $\mathcal{F}_n$ and their total area

Every member of $\mathcal{F}_{n+1}$ is a cornersquare of a $Q \in \mathcal{F}_n$ via the parameter $\varepsilon = \varepsilon_{n+1}$. Let ℓ denote the sidelength of Q. We remove from Q its centersquare εQ. Thus each of the remaining four cornersquares has side-length $\frac{1}{2}(1-\epsilon)\ell$. For $n = 1$ this equals $\frac{1}{2}(1-\epsilon_1)$. Hence, by induction, the side-length of squares in $\mathcal{F}_n$ equals $\frac{1}{2^n}(1-\varepsilon_1)(1-\varepsilon_2)\cdots(1-\varepsilon_n) < \frac{1}{2^n}$. We have 4^n such squares. This sums up to the total area of the union

$$\left|\bigcup \mathcal{F}_n\right| = (1-\varepsilon_1)^2(1-\varepsilon_2)^2\cdots(1-\varepsilon_n)^2$$

The Cantor set We have a decreasing sequence of compact sets $\bigcup \mathcal{F}_1 \supsetneq \bigcup \mathcal{F}_2 \supsetneq \dots \supsetneq \bigcup \mathcal{F}_n \dots$.

Cantor's Theorem tells us that their intersection is not empty,

$$\mathcal{C} \stackrel{\text{def}}{=\!=} \bigcap_{n-1}^{\infty} \left(\bigcup \mathcal{F}_n \right) \neq \emptyset$$

The measure of this Cantor set is positive.

$$|\,\mathcal{C}\,| \;=\; \lim_{n \to \infty} \left| \bigcup \mathcal{F}_n \right| = \prod_{k=1}^{\infty} (1 - \varepsilon_k)^2 \; > 0 \tag{4}$$

The latter inequality is a consequence of $\sum_{k=1}^{\infty} \varepsilon_k < \infty$. Every point in $\mathcal{C}$ is obtained as intersection of exactly one decreasing sequence of the form

$$Q_{\alpha_1}^{\beta_1} \supsetneq Q_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} \supsetneq \; \dots \; \supsetneq Q_{\alpha_1 \alpha_2 \dots \alpha_n}^{\beta_1 \beta_2 \dots \beta_n} \; \dots$$

Another important lemma

Lemma

Every open set that intersects $\mathcal{C}$ *contains a square, say* $Q \in \mathcal{F}_n$ *for sufficiently large* n *which, in turn, contains its centersquare* $\varepsilon_n Q \subset Q$ *.*

Idea: This lemma suggest to consider a monotone mapping $h : \mathbb{Q} \xrightarrow{\text{onto}} \mathbb{Q}$ whose branch set will materialize in the centersquares.

A monotone map $h : \mathbb{Q} \xrightarrow{\text{onto}} \mathbb{Q}$

We let $\mathscr{G}$ denote the family of centersquares of all generations. For every $Q \in \mathscr{G}$ we have a monotone map $h_Q : Q \xrightarrow{\text{onto}} Q$. Recall that h_Q equals the identity map near ∂Q.

Definition

We define the map $h : \mathbb{Q} \xrightarrow{\text{onto}} \mathbb{Q}$ by setting:

$$h(x) = \begin{cases} h_Q(x), & \text{whenever } x \in Q \in \mathscr{G} \\ x\,, & \text{otherwise.} \end{cases} \tag{5}$$

1. $h \in \mathscr{W}^{1,p}(\mathbb{Q}, \mathbb{Q})$ with $p > 2$ and, as such, is continuous on $\overline{\mathbb{Q}}$.
2. For each square (continuum) $Q \in \mathscr{G}$ the mapping $h : Q \xrightarrow{\text{onto}} Q$ is monotone and h is the identity outside those continua.

This is enough to conclude that $h : \overline{\mathbb{Q}} \xrightarrow{\text{onto}} \overline{\mathbb{Q}}$ is monotone.

Every point of the Cantor set $\mathcal{C}$ belongs to the branch set of h. Indeed, by Lemma (intersection), any neighborhood of this point contains a square $Q \in \mathscr{G}$ in which $h = h_Q$ fails to be injective. Thus the branch set $\mathcal{B}_h$ contains the Cantor set $\mathcal{C}$ and, therefore, has positive measure. On the other hand, by the very definition, $h(x) \equiv x$ on $\mathcal{C}$. Therefore $h(\mathcal{B}_h)$ also contains $\mathcal{C}$, so $h(\mathcal{B}_h)$ has positive measure as well.