On a Class of Variational Inequalities

Stanislaw Migórski
Jagiellonian University, Krakow, Poland

A joint work with Dong-ling Cai (Chengdu)
Outline

- Variational-hemivariational inequalities in reflexive Banach spaces
  - Problem formulation and particular cases
  - Existence and uniqueness result
  - Continuous dependence result
  - Existence via a penalty method
  - A generalization: differential variational-hemivariational inequality with history-dependent operators

- Application: a quasistatic unilateral contact problem for viscoplastic material with friction and a nonsmooth multivalued contact condition
Part I

Variational-hemivariational inequalities in reflexive Banach spaces
Problem formulation

Let \((X, \| \cdot \|_X)\) be a reflexive Banach space and \(K \subset X\) be a set. Given an operator \(A : X \to X^*\), functions \(\varphi : K \times K \to \mathbb{R}\), and \(j : X \to \mathbb{R}\), we consider the following problem.

**Problem (1)**

*Find an element \(u \in K\) such that*

\[
\langle Au, v - u \rangle + \varphi(u, v) - \varphi(u, u) + j^0(u; v - u) 
\geq \langle f, v - u \rangle \quad \text{for all} \quad v \in K.
\]

The function \(\varphi(u, \cdot)\) is assumed to be convex and the function \(j\) is locally Lipschitz and, in general, nonconvex. For this reason, inequality in Problem (1) is called a *quasi variational-hemivariational inequality*.
Motivation

The motivation to study Problem (1) comes from the facts:

- various problems considered in the literature can be formulated as Problem (1),
- many problems in mechanics can be formulated in the weak form as Problem (1).
Mathematical tool: convex subdifferential

Let $E$ be a Banach space and $E^*$ be its dual.

Definition (convex subdifferential)

Let $\varphi : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function. The (convex) subdifferential of $\varphi$ at $x$, and is defined by

$$\partial \varphi(x) = \{ x^* \in E^* \mid \varphi(v) \geq \varphi(x) + \langle x^*, v - x \rangle_{E^* \times E} \text{ for all } v \in E \}.$$ 

Sometimes we refer to $\partial \varphi$ as the subdifferential of $\varphi$ in the sense of convex analysis. Observe that if $\varphi(x) = +\infty$, then $\partial \varphi(x) = \emptyset$. 

(Jagiellonian University, Krakow) On a Class of Variational Inequalities
The Clarke subgradient

Definition (Clarke subgradient, 1983)

Let \( h : E \to \mathbb{R} \) be a locally Lipschitz function on a Banach space \( E \).

- **The generalized directional derivative** of \( h \) at \( x \in E \) in the direction \( v \in E \) is defined by
  \[
  h^0(x; v) = \limsup_{y \to x, \ t \downarrow 0} \frac{h(y + tv) - h(y)}{t}.
  \]

- **The generalized subgradient** of \( h \) at \( x \) is given by
  \[
  \partial h(x) = \{ \zeta \in E^* \mid h^0(x; v) \geq \langle \zeta, v \rangle_{E^* \times E} \text{ for all } v \in E \}.
  \]

A locally Lipschitz function \( h \) is called **regular** (in the sense of Clarke) at \( x \in E \) if for all \( v \in E \) the one-sided directional derivative \( h'(x; v) \) exists and satisfies \( h^0(x; v) = h'(x; v) \) for all \( v \in E \).
Particular cases

1. For $j \equiv 0$, Problem (1) reduces to the elliptic quasivariational inequality of the first kind of the form

$$u \in K, \quad \langle Au, v - u \rangle + \varphi(u, v) - \varphi(u, u) \geq \langle f, v - u \rangle \quad \text{for all} \quad v \in K$$

studied, for example, in the book

Particular cases

2. For $j \equiv 0$ and $K = X$, Problem (1) reduces to the elliptic quasivariational inequality of the second kind of the form

$$u \in X, \quad \langle Au, v - u \rangle + \varphi(u, v) - \varphi(u, u) \geq \langle f, v - u \rangle \quad \text{for all } v \in X$$

considered, for example, in the book

Particular cases

3. For \( j \equiv 0 \) and \( \varphi(u, v) = \varphi(v) \), Problem (1) takes the form of the elliptic variational inequality of the first kind of the form

\[
u \in K, \quad \langle Au, v - u \rangle + \varphi(v) - \varphi(u) \geq \langle f, v - u \rangle \quad \text{for all } v \in K\]

treated, for instance, in

Particular cases

4. For \( j \equiv 0, K = X \) and \( \varphi(u, v) = \varphi(v) \), Problem (1) reduces to the elliptic variational inequality of the second kind of the form

\[
  u \in X, \quad \langle Au, v - u \rangle + \varphi(v) - \varphi(u) \geq \langle f, v - u \rangle \quad \text{for all} \quad v \in X
\]

studied, for instance, in


Particular cases

5. For \( j \equiv 0 \) and \( \varphi \equiv 0 \), Problem (1) reduces to the elliptic variational inequality of the form

\[
    u \in K, \quad \langle Au, v - u \rangle \geq \langle f, v - u \rangle \quad \text{for all} \quad v \in K
\]

considered, for instance, in


Particular cases

6. For $\varphi \equiv 0$, $K = X$, from Problem (1), we obtain the elliptic hemivariational inequality of the form

$$u \in X, \quad \langle Au, v \rangle + j^0(u; v) \geq \langle f, v \rangle \quad \text{for all} \quad v \in X$$

investigated, for instance, in


Particular cases

7. For $j \equiv 0$, $\varphi \equiv 0$ and $K = X$, Problem (1) reduces to the elliptic equation

$$u \in X, \; Au = f.$$
Particular cases

8. For \( K = X = V \) and \( \varphi(u, v) = \int_{\Gamma} (Fu)\theta(\gamma v) \, d\Gamma \) for \( u, v \in V \),
Problem (1) reduces to: find \( u \in V \) such that

\[
\langle Au, v - u \rangle + \int_{\Gamma} (Fu) (\theta(\gamma v) - \theta(\gamma u)) \, d\Gamma \\
+ j^0(u; v - u) \geq \langle f, v - u \rangle \quad \text{for all } v \in V
\]

and it was studied in


Here \( \Gamma \subseteq \partial \Omega \) is a measurable part of the boundary of an open bounded subset \( \Omega \) of \( \mathbb{R}^d \), \( V \) is a closed subspace of \( H^1(\Omega; \mathbb{R}^s) \), \( F: V \to L^2(\Gamma) \) and \( \theta: \mathbb{R}^s \to \mathbb{R} \) are Lipschitz continuous \( Fv \geq 0 \) for all \( v \in V \), \( \theta \) is convex, and \( \gamma: V \to L^2(\Gamma; \mathbb{R}^s) \) denotes the trace operator.
Hypotheses on the data of Problem (1)

\(H(K)\): \(K\) is a nonempty, closed and convex subset of \(X\).

\(H(f)\): \(f \in X^*\).

\(H(A)\):

\[
A: X \to X^* \text{ is such that}
\]

(a) it is pseudomonotone: it is bounded, and \(u_n \to u\) weakly in \(X\) with
\[
\lim \sup \langle Au_n, u_n - u \rangle \leq 0
\]
implies \(\lim \langle Au_n, u_n - u \rangle = 0\) and \(Au_n \to Au\) weakly in \(X^*\).

(b) there exist \(\alpha_A > 0, \beta, \gamma \in \mathbb{R}\) and \(u_0 \in K\) such that
\[
\langle Av, v - u_0 \rangle \geq \alpha_A \|v\|_X^2 - \beta \|v\|_X - \gamma \quad \text{for all } \ v \in X.
\]

(c) strongly monotone, i.e., there exists \(m_A > 0\) such that
\[
\langle Av_1 - Av_2, v_1 - v_2 \rangle \geq m_A \|v_1 - v_2\|_X^2 \quad \text{for all } \ v_1, v_2 \in X.
\]
Hypothesis \( H(\varphi) \)

\[
\begin{align*}
\varphi: & \ K \times K \to \mathbb{R} \text{ is such that} \\
(a) & \ \varphi(\eta, \cdot): K \to \mathbb{R} \text{ is convex and lower semicontinuous on } K, \\
& \text{for all } \eta \in K. \\
(b) & \text{there exists } \alpha_\varphi > 0 \text{ such that} \\
& \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \\
& \leq \alpha_\varphi \|\eta_1 - \eta_2\|_X \|v_1 - v_2\|_X \\
& \text{for all } \eta_1, \eta_2, v_1, v_2 \in K.
\end{align*}
\]
Hypothesis $H(j)$

\[
\begin{align*}
j: X \to \mathbb{R} & \text{ is such that} \\
\text{(a) } j & \text{ is locally Lipschitz.} \\
\text{(b) } \|\partial j(v)\|_{X^*} & \leq c_0 + c_1 \|v\|_X \text{ for all } v \in X \text{ with } c_0, c_1 \geq 0. \\
\text{(c) } & \text{there exists } \alpha_j \geq 0 \text{ such that} \\
& j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2 \\
& \text{for all } v_1, v_2 \in X.
\end{align*}
\]
Existence and uniqueness result

Theorem (2)

Assume $H(K)$, $H(f)$, $H(A)$, $H(\varphi)$, $H(j)$ and, in addition, assume the smallness conditions

\[ \alpha \varphi + \alpha j < m_A, \]
\[ \alpha j < \alpha A. \]

Then, Problem (1) has a unique solution $u \in K$. 
Remark on hypothesis $H(j)$

- If $j: X \rightarrow \mathbb{R}$ is a locally Lipschitz function, then hypothesis $H(j)(c)$:
  \[
  j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2 \quad \text{for all } v_1, v_2 \in X
  \]
  with $\alpha_j \geq 0$, is equivalent to
  \[
  \langle \partial j(v_1) - \partial j(v_2), v_1 - v_2 \rangle \geq -\alpha_j \|v_1 - v_2\|_X^2 \quad \text{for all } v_1, v_2 \in X. \quad (1)
  \]
  The latter is called the relaxed monotonicity condition.

- Note also that if $j: X \rightarrow \mathbb{R}$ is a convex function, then $H(j)(c)$ or, equivalently, condition (1) always holds since it reduces to the monotonicity of the (convex) subdifferential, i.e., $\alpha_j = 0$. 
Remark on $H(j)$

- A convex and continuous function $f : X \to \mathbb{R}$ is locally Lipschitz. More generally, a convex function $f : X \to \mathbb{R}$, which is bounded above on a neighborhood of some point is locally Lipschitz (see Clarke).

- A function $f : X \to \mathbb{R}$, which is Lipschitz continuous on bounded subsets of $X$ is locally Lipschitz. The converse assertion is not generally true.

Continuous dependence result for Problem (1)

Let $\rho > 0$ be a parameter. Consider the following version of Problem (1).

\begin{itemize}
  \item \textbf{Problem (1$_\rho$)}
  \item \textit{Find} $u_\rho \in K$ \textit{such that}
  \begin{align*}
  \langle Au_\rho, v - u_\rho \rangle + \varphi_\rho(u_\rho, v) - \varphi_\rho(u_\rho, u_\rho) + j_\rho^0(u_\rho; v - u_\rho) \\
  \geq \langle f_\rho, v - u_\rho \rangle \quad \text{for all} \quad v \in K.
  \end{align*}
\end{itemize}

Consider the functions $\varphi_\rho$, $j_\rho$ and $f_\rho$ which satisfy hypotheses $H(\varphi)$, $H(j)$ and $H(f)$ with constants $\alpha_{\varphi_\rho}$ and $\alpha_{j_\rho}$, respectively. Assume that there exists $m_0 \in \mathbb{R}$ such that $\alpha_{\varphi_\rho} + \alpha_{j_\rho} \leq m_0 < m_A$ for all $\rho > 0$, and $\alpha_{j_\rho} < \alpha_A$ for all $\rho > 0$.

Theorem (2) guarantees that Problem (1$_\rho$) has a unique solution $u_\rho \in K$, for each $\rho > 0$. 
Continuous dependence result for Problem (1)

We now consider the following hypotheses.

\[
\begin{align*}
\text{There exists a function } \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } g \in \mathbb{R}_+ \text{ such that } \\
\varphi(\eta, v) - \varphi(\eta, \eta) - \varphi_{\rho}(\eta, v) + \varphi_{\rho}(\eta, \eta) & \leq G(\rho)(\|\eta\|_X + g)\|\eta - v\|_X \\
\text{for all } \eta, v \in X, \rho > 0 \text{ and } \lim_{\rho \to 0} G(\rho) = 0.
\end{align*}
\]

\[
\begin{align*}
\text{There exists a function } H : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } h \in \mathbb{R}_+ \text{ such that } \\
\varphi(\eta, v) - \varphi(\eta, \eta) - \varphi_{\rho}(\eta, v) + \varphi_{\rho}(\eta, \eta) & \leq H(\rho)(\|\eta\|_X + h)\|\eta - v\|_X \\
\text{for all } \eta, v \in X, \rho > 0 \text{ and } \lim_{\rho \to 0} H(\rho) = 0.
\end{align*}
\]

\[
f_{\rho} \to f \text{ in } X^*, \text{ as } \rho \to 0.
\]

**Theorem**

*Assume the hypotheses above. Then* \( u_{\rho} \to u \text{ in } X, \text{ as } \rho \to 0. *
A penalty method – formulation

We can prove the existence and uniqueness of solution to the variational-hemivariational inequality by applying a penalty method. We consider the following problem.

Problem (2)

Find an element \( u \in K \) such that

\[
\langle Au, v - u \rangle + \varphi(v) - \varphi(u) + j^0(u; v - u) \geq \langle f, v - u \rangle \quad \text{for all} \quad v \in K.
\]

Note that Problem (2) is a particular case of Problem (1) obtained for the function \( \varphi \) independent of the first variable. We need the following additional hypotheses.

\[
\begin{align*}
\varphi & : X \to \mathbb{R} \text{ is convex and lower semicontinuous} \quad (2) \\
\left\{ 
\begin{array}{l}
\text{\( j : X \to \mathbb{R} \) is such that} \\
\limsup j^0(u_n; v - u_n) \leq j^0(u; v - u)
\end{array}
\right. \\
\text{for all} \quad v \in X \quad \text{and} \quad u_n \to u \text{ weakly in} \ X. \quad (3)
\end{align*}
\]
A penalty method – formulation

We adopt the following notion of the penalty operator.

**Definition (a penalty operator)**

A single-valued operator $P: X \rightarrow X^*$ is said to be a *penalty operator* of $K$ if $P$ is bounded, demicontinuous, monotone and $K = \{ x \in X \mid Px = 0 \}$.

Then, for every $\lambda > 0$, we consider the following penalized problem.

**Problem (3)**

*Find an element $u_\lambda \in X$ such that*

$$
\langle Au_\lambda, v - u_\lambda \rangle + \frac{1}{\lambda} \langle Pu_\lambda, v - u_\lambda \rangle + \varphi(v) - \varphi(u_\lambda) + j^0(u_\lambda; v - u_\lambda) \geq \langle f, v - u_\lambda \rangle
$$

*for all $v \in X$, where $P: X \rightarrow X^*$ is the penalty operator of $K$.***
A penalty method – convergence

Our main result for the penalty method is the following.

**Theorem**

Assume the hypotheses above, let $P$ be a penalty operator of $K$, and

$$\alpha_j < \min \{\alpha_A, m_A\}.$$  

Then

(i) for each $\lambda > 0$, there exists a unique solution $u_\lambda \in X$ to Problem (3);

(ii) $u_\lambda \to u$ in $X$, as $\lambda \to 0$, where $u \in K$ is a unique solution to Problem (2).
Example

We provide sufficient conditions for functions which satisfy our hypotheses $H(j)$ and (3).

Lemma

Let $X$ and $Y$ be reflexive Banach spaces, $\psi : Y \to \mathbb{R}$ be a function which satisfies $H(j)$ and $\psi$ (or $-\psi$) is regular, and let $M : X \to Y$ be given by

$$Mv = Lv + v_0,$$

where $L : X \to Y$ is a linear compact operator and $v_0 \in Y$ is fixed. Define the function $j : X \to \mathbb{R}$ by

$$j(v) = \psi(Mv) \quad \text{for} \quad v \in X.$$

Then the function $j$ satisfies conditions $H(j)$ and (3).
Differential variational-hemivariational inequality with history-dependent operators

Problem

Find \( x \in H^1(0, T; U) \) and \( w \in L^2(0, T; K) \) such that

\[
\begin{cases}
  x'(t) = F(t, x(t), w(t), (R_0 w)(t)) \quad \text{for a.e. } t \in (0, T), \\
  \langle A(t, x(t), (R_1 w)(t), w(t)) - f(t, (R_2 w)(t)), v - w(t) \rangle \\
  + j^0(t, x(t), (R_4 w)(t), Mw(t); Mv - Mw(t)) \\
  + \varphi(t, x(t), (R_3 w)(t), w(t), v) - \varphi(t, x(t), (R_3 w)(t), w(t), w(t)) \geq 0 \\
  \text{for all } v \in K, \text{ a.e. } t \in (0, T), \\
  x(0) = x_0.
\end{cases}
\]
Definition

Given normed spaces $\mathbb{X}$ and $\mathbb{Y}$, we say that an operator

$$S: L^2(0, T; \mathbb{X}) \rightarrow L^2(0, T; \mathbb{Y})$$

is a history-dependent operator, if there is a constant $c > 0$ such that

$$\|(Sv_1)(t) - (Sv_2)(t)\|_{\mathbb{Y}} \leq c \int_0^t \|v_1(s) - v_2(s)\|_{\mathbb{X}} ds$$

for all $v_1, v_2 \in L^2(0, T; \mathbb{X})$, a.e. $t \in (0, T)$. 
Part II

An application to a quasistatic contact problem

A quasistatic unilateral contact problem for viscoplastic material with friction and a non-smooth multivalued contact condition.

The linearized strain-displacement relation are given by

\[ \varepsilon_{ij}(u) = (\varepsilon(u))_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in} \quad \Omega, \]

where \( u_{i,j} = \partial u_i / \partial x_j \).
Find a displacement $u : Q \to \mathbb{R}^d$, a stress $\sigma : Q \to \mathbb{S}^d$, and the adhesion $\beta : \Gamma_C \times (0, T) \to [0, 1]$ such that for all $t \in (0, T)$, $u(0) = u_0$, $\beta(0) = \beta_0$

\[
0 = \text{Div}\sigma(t) + f_0(t) \quad \text{in} \quad \Omega,
\]
\[
\sigma(t) = A(t, \varepsilon(u'(t))) + B(t, \varepsilon(u(t))) + \int_0^t G(s, \sigma(s) - A(s, \varepsilon(u'(s))), \varepsilon(u(s))) \, ds \quad \text{in} \quad \Omega,
\]
\[
u(t) = 0 \quad \text{on} \quad \Gamma_D,
\]
\[
\sigma(t)\nu = f_N(t) \quad \text{on} \quad \Gamma_N,
\]
\[
\sigma_{\nu}(t) = \sigma_{\nu}^1(t) + \sigma_{\nu}^2(t), \quad -\sigma_{\nu}^1(t) \in p_{\nu}(t, \beta, u_{\nu}(t)) \partial j(u_{\nu}'(t)) \quad \text{on} \quad \Gamma_C,
\]
\[
u'(t) \leq g, \quad \sigma_{\nu}^2(t) + p(t, \beta(t), u_{\nu}(t), u_{\nu}'(t)) \leq 0,
\]
\[
(u_{\nu}'(t) - g)(\sigma_{\nu}^2(t) + p(t, \beta(t), u_{\nu}(t), u_{\nu}'(t))) = 0 \quad \text{on} \quad \Gamma_C,
\]
\[
\|\sigma_{\tau}(t)\| \leq \mu p(t, \beta(t), u_{\nu}(t), u_{\nu}'(t)) \quad \text{on} \quad \Gamma_C,
\]
\[
-\sigma_{\tau}(t) = \mu p(t, \beta(t), u_{\nu}(t), u_{\nu}'(t)) \frac{u_{\tau}'(t)}{\|u_{\tau}'(t)\|}, \quad \text{if} \quad u_{\tau}'(t) \neq 0 \quad \text{on} \quad \Gamma_C,
\]
\[
\beta'(t) = h(t, u_{\nu}(t), u_{\tau}(t), \beta(t)) \quad \text{on} \quad \Gamma_C.
\]
Example: nonconvex friction law

\[ j(\xi) = \max\{f_1(\xi), f_2(\xi)\} \]

Clarke subgradient \( \partial j \)
Example: nonconvex friction law

\[ j(\xi) = \max\{a\|\xi\|, f_1(\xi), f_2(\xi)\} \]

Clarke subgradient \( \partial j \)
Example: zig-zag friction law

\[ j(\xi) = \max\{ f_1(\xi), f_2(\xi), f_3(\xi), f_1'(\xi), f_2'(\xi), f_3'(\xi) \} \]

Clarke subgradient \( \partial j \)
Example: zig-zag friction law

\[ j(\xi) = \min\{f_1(\xi), f_2(\xi), f_3(\xi)\} \]

Clarke subgradient \( \partial j \)
Example: infinite number of jumps

Let $I$ be an open subset of the real line $\mathbb{R}$ and let $M$ be a measurable subset of $I$ such that for every open and nonempty subset $I$ of $I$, $\text{meas}(I \cap M) > 0$ and $\text{meas}(I \cap (I \setminus M)) > 0$. Let

$$g(s) = \begin{cases} b_1 & \text{if } s \in M \\ -b_2 & \text{if } s \notin M \end{cases}$$

and $j(r) = \int_0^r g(\theta) \, d\theta$. Then the nonconvex potential $j$ is locally Lipschitz and

$$\partial j(r) = [-b_2, b_1] \text{ for every } r \in I.$$
Monographs

2013

Nonlinear Inclusions and Hemivariational Inequalities
Models and Analysis of Contact Problems

2015

Variational-Hemivariational Inequalities with Applications

2018

Thank you very much for your attention!