On a Class of Variational Inequalities

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Outline

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  - Problem formulation and particular cases
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  - Continuous dependence result
  - Existence via a penalty method
  - A generalization: differential variational-hemivariational inequality with history-dependent operators
- Application: a quasistatic unilateral contact problem for viscoplastic material with friction and a nonsmooth multivalued contact condition

Part I

Variational-hemivariational inequalities in reflexive Banach spaces

Problem formulation

Let $(X, \|\cdot\|_X)$ be a reflexive Banach space and $K \subset X$ be a set. Given an operator $A\colon X \to X^*$, functions $\varphi\colon K \times K \to \mathbb{R}$, and $j\colon X \to \mathbb{R}$, we consider the following problem.

Problem (1)

Find an element $u \in K$ such that

$$\langle Au, v-u\rangle + \varphi(u,v) - \varphi(u,u) + j^0(u; v-u) \geq \langle f, v-u\rangle \quad \text{for all} \quad v \in K.$$

The function $\varphi(u,\cdot)$ is assumed to be convex and the function j is locally Lipschitz and, in general, nonconvex. For this reason, inequality in Problem (1) is called a *quasi variational-hemivariational inequality*.

Motivation

The motivation to study Problem (1) comes from the facts:

- various problems considered in the literature can be formulated as Problem (1),
- many problems in mechanics can be formulated in the weak form as Problem (1).

Mathematical tool: convex subdifferential

Let E be a Banach space and E^* be its dual.

Definition (convex subdifferential)

Let $\varphi \colon E \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The (convex) subdifferential of φ at x, and is defined by

$$\partial\varphi(x) = \{\, x^* \in E^* \mid \varphi(v) \geq \varphi(x) + \langle x^*, v - x\rangle_{E^* \times E} \ \text{ for all } \ v \in E \,\}.$$

Sometimes we refer to $\partial\varphi$ as the subdifferential of φ in the sense of convex analysis. Observe that if $\varphi(x) = +\infty$, then $\partial\varphi(x) = \emptyset$.

The Clarke subgradient

Definition (Clarke subgradient, 1983)

Let $h\colon E \to \mathbb{R}$ be a locally Lipschitz function on a Banach space E.

- The generalized directional derivative of h at $x \in E$ in the direction $v \in E$ is defined by

$$h^0(x; v) = \limsup_{y \to x,\ t \downarrow 0} \frac{h(y + tv) - h(y)}{t}.$$

- The generalized subgradient of h at x is given by

$$\partial h(x) = \{\, \zeta \in E^* \mid h^0(x; v) \geq \langle \zeta, v \rangle_{E^* \times E} \text{ for all } v \in E \,\}.$$

A locally Lipschitz function h is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h'(x; v)$ exists and satisfies $h^0(x; v) = h'(x; v)$ for all $v \in E$.

Particular cases

1. For $j \equiv 0$, Problem (1) reduces to the elliptic quasivariational inequality of the first kind of the form

$$u \in K, \quad \langle Au, v - u \rangle + \varphi(u, v) - \varphi(u, u) \geq \langle f, v - u \rangle \text{ for all } v \in K$$

studied, for example, in the book

M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Note Series **398**, Cambridge University Press, 2012.

Particular cases

2. For $j \equiv 0$ and $K = X$, Problem (1) reduces to the elliptic quasivariational inequality of the second kind of the form

$$u \in X, \quad \langle Au, v - u \rangle + \varphi(u, v) - \varphi(u, u) \geq \langle f, v - u \rangle \ \text{ for all } \ v \in X$$

considered, for example, in the book

M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Note Series **398**, Cambridge University Press, 2012.

Particular cases

3. For $j \equiv 0$ and $\varphi(u, v) = \varphi(v)$, Problem (1) takes the form of the elliptic variational inequality of the first kind of the form

$$u \in K, \quad \langle Au, v - u \rangle + \varphi(v) - \varphi(u) \geq \langle f, v - u \rangle \ \text{ for all } \ v \in K$$

treated, for instance, in

J.L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* **20** (1967), 493–519.

Particular cases

4. For $j \equiv 0$, $K = X$ and $\varphi(u,v) = \varphi(v)$, Problem (1) reduces to the elliptic variational inequality of the second kind of the form

$$u \in X, \quad \langle Au, v-u\rangle + \varphi(v) - \varphi(u) \geq \langle f, v-u\rangle \ \text{ for all } \ v \in X$$

studied, for instance, in

H. Brézis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier (Grenoble)* **18** (1968), 115–175.

J.L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* **20** (1967), 493–519.

M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Note Series **398**, Cambridge University Press, 2012.

Particular cases

5. For $j \equiv 0$ and $\varphi \equiv 0$, Problem (1) reduces to the elliptic variational inequality of the form

$$u \in K, \quad \langle Au, v - u \rangle \geq \langle f, v - u \rangle \ \text{ for all } \ v \in K$$

considered, for instance, in

H. Brézis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier (Grenoble)* **18** (1968), 115–175.

F.E. Browder, Nonlinear monotone operators and convex sets in Banach spaces, *Bull. Amer. Math. Soc.* **71** (1965), 780–785.

Particular cases

6. For $\varphi \equiv 0$, $K = X$, from Problem (1), we obtain the elliptic hemivariational inequality of the form

$$u \in X, \quad \langle Au, v \rangle + j^0(u; v) \geq \langle f, v \rangle \ \text{ for all } \ v \in X$$

investigated, for instance, in

Z. Naniewicz and P. D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.

S. Migorski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics **26**, Springer, New York, 2013.

Particular cases

7. For $j \equiv 0$, $\varphi \equiv 0$ and $K = X$, Problem (1) reduces to the elliptic equation

$$u \in X, \; Au = f.$$

Particular cases

8. For $K = X = V$ and $\varphi(u, v) = \int_\Gamma (Fu)\theta(\gamma v)\, d\Gamma$ for u, $v \in V$, Problem (1) reduces to: find $u \in V$ such that

$$\langle Au, v - u\rangle + \int_\Gamma (Fu)\,(\theta(\gamma v) - \theta(\gamma u))\; d\Gamma$$
$$+ j^0(u; v - u) \geq \langle f, v - u\rangle \quad \text{for all } v \in V$$

and it was studied in

W. Han, S. Migórski, M. Sofonea, A class of variational-hemivariational inequalities with applications to frictional contact problems, *SIAM Journal of Mathematical Analysis* **46** (2014), 3891–3912.

Here $\Gamma \subseteq \partial\Omega$ is a measurable part of the boundary of an open bounded subset Ω of $\mathbb{R}^d$, V is a closed subspace of $H^1(\Omega; \mathbb{R}^s)$, $F\colon V \to L^2(\Gamma)$ and $\theta\colon \mathbb{R}^s \to \mathbb{R}$ are Lipschitz continuous $Fv \geq 0$ for all $v \in V$, θ is convex, and $\gamma\colon V \to L^2(\Gamma; \mathbb{R}^s)$ denotes the trace operator.

Hypotheses on the data of Problem (1)

$H(K)$: K is a nonempty, closed and convex subset of X.

$H(f)$: $f \in X^*$.

$H(A)$:

$$\begin{cases} A\colon X \to X^* \text{ is such that} \\ \text{(a) it is pseudomonotone : it is bounded, and } u_n \to u \text{ weakly in } X \\ \quad \text{with } \limsup \langle Au_n, u_n - u\rangle \le 0 \text{ imply } \lim \langle Au_n, u_n - u\rangle = 0 \\ \quad \text{and } Au_n \to Au \text{ weakly in } X^*. \\ \text{(b) there exist } \alpha_A > 0, \beta, \gamma \in \mathbb{R} \text{ and } u_0 \in K \text{ such that} \\ \qquad \langle Av, v - u_0\rangle \ge \alpha_A \|v\|_X^2 - \beta \|v\|_X - \gamma \quad \text{for all } v \in X. \\ \text{(c) strongly monotone, i.e., there exists } m_A > 0 \text{ such that} \\ \qquad \langle Av_1 - Av_2, v_1 - v_2\rangle \ge m_A \|v_1 - v_2\|_X^2 \quad \text{for all } v_1, v_2 \in X. \end{cases}$$

Hypothesis $H(\varphi)$

$$
\begin{cases}
\varphi\colon K \times K \to \mathbb{R} \text{ is such that} \\
\text{(a) } \varphi(\eta, \cdot)\colon K \to \mathbb{R} \text{ is convex and lower semicontinuous on } K, \\
\qquad \text{for all } \eta \in K. \\
\text{(b) there exists } \alpha_\varphi > 0 \text{ such that} \\
\qquad \varphi(\eta_1, v_2) - \varphi(\eta_1, v_1) + \varphi(\eta_2, v_1) - \varphi(\eta_2, v_2) \\
\qquad\qquad \le \alpha_\varphi \|\eta_1 - \eta_2\|_X \, \|v_1 - v_2\|_X \\
\qquad \text{for all } \eta_1, \eta_2, v_1, v_2 \in K.
\end{cases}
$$

Hypothesis $H(j)$

$$\begin{cases} j\colon X \to \mathbb{R} \text{ is such that} \\ \text{(a) } j \text{ is locally Lipschitz.} \\ \text{(b) } \|\partial j(v)\|_{X^*} \le c_0 + c_1 \|v\|_X \text{ for all } v \in X \text{ with } c_0, c_1 \ge 0. \\ \text{(c) there exists } \alpha_j \ge 0 \text{ such that} \\ \qquad j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \le \alpha_j \|v_1 - v_2\|_X^2 \\ \qquad \text{for all } v_1, v_2 \in X. \end{cases}$$

Existence and uniqueness result

Theorem (2)

Assume $H(K)$, $H(f)$, $H(A)$, $H(\varphi)$, $H(j)$ and, in addition, assume the smallness conditions

$$\alpha_\varphi + \alpha_j < m_A,$$

$$\alpha_j < \alpha_A.$$

Then, Problem (1) *has a unique solution $u \in K$.*

Remark on hypothesis $H(j)$

- If $j\colon X \to \mathbb{R}$ is a locally Lipschitz function, then hypothesis $H(j)$(c):

  $$j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \le \alpha_j \|v_1 - v_2\|_X^2 \quad \text{for all} \quad v_1, v_2 \in X$$

  with $\alpha_j \ge 0$, is equivalent to

  $$\langle \partial j(v_1) - \partial j(v_2), v_1 - v_2 \rangle \ge -\alpha_j \|v_1 - v_2\|_X^2 \quad \text{for all} \quad v_1, v_2 \in X. \quad (1)$$

  The latter is called the relaxed monotonicity condition.

- Note also that if $j\colon X \to \mathbb{R}$ is a convex function, then $H(j)$(c) or, equivalently, condition (1) always holds since it reduces to the monotonicity of the (convex) subdifferential, i.e., $\alpha_j = 0$.

Remark on $H(j)$

- A convex and continuous function $f\colon X \to \mathbb{R}$ is locally Lipschitz. More generally, a convex function $f\colon X \to \mathbb{R}$, which is bounded above on a neighborhood of some point is locally Lipschitz (see Clarke).
- A function $f\colon X \to \mathbb{R}$, which is Lipschitz continuous on bounded subsets of X is locally Lipschitz. The converse assertion is not generally true.

F. H. Clarke, *Optimization and Nonsmooth Analysis*, Society for Industrial and Applied Mathematics, Philadelphia, PA (1990).

Continuous dependence result for Problem (1)

Let $\rho > 0$ be a parameter. Consider the following version of Problem (1).

Problem (1_ρ)

Find $u_\rho \in K$ such that

$$\langle Au_\rho, v - u_\rho\rangle + \varphi_\rho(u_\rho, v) - \varphi_\rho(u_\rho, u_\rho) + j_\rho^0(u_\rho; v - u_\rho)$$
$$\geq \langle f_\rho, v - u_\rho\rangle \ \text{ for all } \ v \in K.$$

Consider the functions φ_ρ, j_ρ and f_ρ which satisfy hypotheses $H(\varphi)$, $H(j)$ and $H(f)$ with constants α_{φ_ρ} and α_{j_ρ}, respectively. Assume that

$$\text{there exists } m_0 \in \mathbb{R} \text{ such that } \alpha_{\varphi_\rho} + \alpha_{j_\rho} \leq m_0 < m_A \ \text{ for all } \ \rho > 0,$$

$$\text{and } \ \alpha_{j_\rho} < \alpha_A \ \text{ for all } \ \rho > 0.$$

Theorem (2) guarantees that Problem (1_ρ) has a unique solution $u_\rho \in K$, for each $\rho > 0$.

Continuous dependence result for Problem (1)

We now consider the following hypotheses.

$$\begin{cases} \text{There exists a function } G\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } g \in \mathbb{R}_+ \text{ such that} \\ \varphi(\eta, v) - \varphi(\eta, \eta) - \varphi_\rho(\eta, v) + \varphi_\rho(\eta, \eta) \leq G(\rho)(\|\eta\|_X + g)\|\eta - v\|_X \\ \text{for all } \eta, v \in X,\ \rho > 0 \text{ and } \lim_{\rho \to 0} G(\rho) = 0. \end{cases}$$

$$\begin{cases} \text{There exists a function } H\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } h \in \mathbb{R}_+ \text{ such that} \\ \quad j^0(u; v) - j^0_\rho(u; v) \leq H(\rho)(\|u\|_X + h)\|u - v\|_X \\ \text{for all } u, v \in X,\ \rho > 0 \text{ and } \lim_{\rho \to 0} H(\rho) = 0. \end{cases}$$

$$f_\rho \to f \quad \text{in} \quad X^*, \quad \text{as } \rho \to 0.$$

Theorem

Assume the hypotheses above. Then $u_\rho \to u$ in X, as $\rho \to 0$.

A penalty method – formulation

We can prove the existence and uniqueness of solution to the variational-hemivariational inequality by applying a penalty method. We consider the following problem.

Problem (2)

Find an element $u \in K$ such that

$$\langle Au, v - u \rangle + \varphi(v) - \varphi(u) + j^0(u; v - u) \geq \langle f, v - u \rangle \quad \text{for all} \quad v \in K.$$

Note that Problem (2) is a particular case of Problem (1) obtained for the function φ independent of the first variable. We need the following additional hypotheses.

$$\varphi \colon X \to \mathbb{R} \text{ is convex and lower semicontinuous} \tag{2}$$

$$\begin{cases} j \colon X \to \mathbb{R} \text{ is such that } \limsup j^0(u_n; v - u_n) \leq j^0(u; v - u) \\ \text{for all } v \in X \text{ and } u_n \to u \text{ weakly in } X. \end{cases} \tag{3}$$

A penalty method – formulation

We adopt the following notion of the penalty operator.

Definition (a penalty operator)

A single-valued operator $P\colon X \to X^*$ is said to be *a penalty operator* of K if P is bounded, demicontinuous, monotone and $K = \{x \in X \mid Px = 0\}$.

Then, for every $\lambda > 0$, we consider the following penalized problem.

Problem (3)

Find an element $u_\lambda \in X$ such that

$$\langle Au_\lambda, v - u_\lambda \rangle + \frac{1}{\lambda}\langle Pu_\lambda, v - u_\lambda \rangle + \varphi(v) - \varphi(u_\lambda) + j^0(u_\lambda; v - u_\lambda) \geq \langle f, v - u_\lambda \rangle$$

for all $v \in X$, where $P\colon X \to X^*$ is the penalty operator of K.

A penalty method – convergence

Our main result for the penalty method is the following.

Theorem

Assume the hypotheses above, let P be a penalty operator of K, and

$$\alpha_j < \min\{\alpha_A, m_A\}.$$

Then

(i) *for each $\lambda > 0$, there exists a unique solution $u_\lambda \in X$ to Problem (3);*

(ii) *$u_\lambda \to u$ in X, as $\lambda \to 0$, where $u \in K$ is a unique solution to Problem (2).*

Example

We provide sufficient conditions for functions which satisfy our hypotheses $H(j)$ and (3).

Lemma

Let X and Y be reflexive Banach spaces, $\psi\colon Y \to \mathbb{R}$ be a function which satisfies $H(j)$ and ψ (or $-\psi$) is regular, and let $M\colon X \to Y$ be given by

$$Mv = Lv + v_0,$$

where $L\colon X \to Y$ is a linear compact operator and $v_0 \in Y$ is fixed. Define the function $j\colon X \to \mathbb{R}$ by

$$j(v) = \psi(Mv) \quad \text{for} \quad v \in X.$$

Then the function j satisfies conditions $H(j)$ and (3).

Differential variational-hemivariational inequality with history-dependent operators

Problem

Find $\mathrm{x} \in H^1(0, T; U)$ *and* $w \in L^2(0, T; K)$ *such that*

$$\begin{cases} \mathrm{x}'(t) = F(t, \mathrm{x}(t), w(t), (R_0 w)(t)) \ \text{ for a.e. } \ t \in (0, T), \\ \langle A(t, \mathrm{x}(t), (R_1 w)(t), w(t)) - f(t, (R_2 w)(t)), v - w(t) \rangle \\ \quad + j^0(t, \mathrm{x}(t), (R_4 w)(t), Mw(t); Mv - Mw(t)) \\ \quad + \varphi(t, \mathrm{x}(t), (R_3 w)(t), w(t), v) - \varphi(t, \mathrm{x}(t), (R_3 w)(t), w(t), w(t)) \geq 0 \\ \qquad\qquad \text{for all } \ v \in K, \ \text{a.e. } t \in (0, T), \\ \mathrm{x}(0) = \mathrm{x}_0. \end{cases}$$

Definition

Given normed spaces $\mathbb{X}$ and $\mathbb{Y}$, we say that an operator

$$\mathbb{S}\colon L^2(0, T; \mathbb{X}) \to L^2(0, T; \mathbb{Y})$$

is a history-dependent operator, if there is a constant $c > 0$ such that

$$\|(\mathbb{S}v_1)(t) - (\mathbb{S}v_2)(t)\|_{\mathbb{Y}} \leq c \int_0^t \|v_1(s) - v_2(s)\|_{\mathbb{X}}\, ds$$

for all $v_1, v_2 \in L^2(0, T; \mathbb{X})$, a.e. $t \in (0, T)$.

Part II

An application to a quasistatic contact problem

A quasistatic unilateral contact problem for viscoplastic material with friction and a nonsmooth multivalued contact condition.

The linearized strain-displacement relation are given by

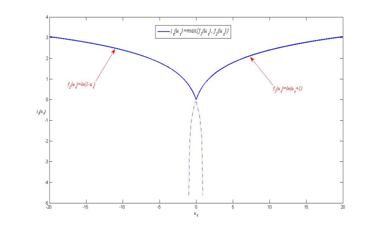
$$\varepsilon_{ij}(\boldsymbol{u}) = (\varepsilon(\boldsymbol{u}))_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \ \text{ in } \ \Omega,$$

where $u_{i,j} = \partial u_i / \partial x_j$.

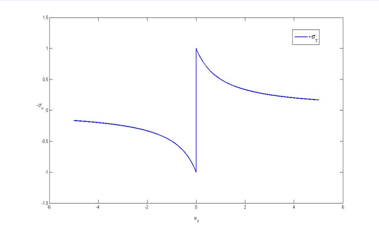
Find a displacement $\boldsymbol{u}\colon Q \to \mathbb{R}^d$, a stress $\boldsymbol{\sigma}\colon Q \to \mathbb{S}^d$, and the adhesion $\beta\colon \Gamma_C \times (0,T) \to [0,1]$ such that for all $t \in (0,T)$, $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\beta(0) = \beta_0$

$$
\begin{aligned}
\mathbf{0} &= \mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) && \text{in} \quad \Omega,\\
\boldsymbol{\sigma}(t) &= A(t,\varepsilon(\boldsymbol{u}'(t))) + B(t,\varepsilon(\boldsymbol{u}(t)))\\
&\qquad + \int_0^t G\big(s,\boldsymbol{\sigma}(s) - A(s,\varepsilon(\boldsymbol{u}'(s))),\varepsilon(\boldsymbol{u}(s)\big)\,ds && \text{in} \quad \Omega,\\
\boldsymbol{u}(t) &= \mathbf{0} && \text{on} \quad \Gamma_D,\\
\boldsymbol{\sigma}(t)\boldsymbol{\nu} &= \boldsymbol{f}_N(t) && \text{on} \quad \Gamma_N,\\
\sigma_\nu(t) &= \sigma_\nu^1(t) + \sigma_\nu^2(t),\ \ -\sigma_\nu^1(t) \in p_\nu(t,\beta,u_\nu(t))\,\partial j(u_\nu'(t)) && \text{on} \quad \Gamma_C,\\
u_\nu'(t) &\le g,\ \ \sigma_\nu^2(t) + p(t,\beta(t),u_\nu(t),u_\nu'(t)) \le 0,\\
&\qquad (u_\nu'(t) - g)(\sigma_\nu^2(t) + p(t,\beta(t),u_\nu(t),u_\nu'(t))) = 0 && \text{on} \quad \Gamma_C,\\
\|\boldsymbol{\sigma}_\tau(t)\| &\le \mu\, p(t,\beta(t),u_\nu(t),u_\nu'(t)) && \text{on} \quad \Gamma_C,\\
-\,\boldsymbol{\sigma}_\tau(t) &= \mu\, p(t,\beta(t),u_\nu(t),u_\nu'(t))\frac{\boldsymbol{u}_\tau'(t)}{\|\boldsymbol{u}_\tau'(t)\|}, \quad \text{if } \boldsymbol{u}_\tau'(t) \neq 0 && \text{on} \quad \Gamma_C,\\
\beta'(t) &= h(t,u_\nu(t),\boldsymbol{u}_\tau(t),\beta(t)), && \text{on} \quad \Gamma_C.
\end{aligned}
$$

Example: nonconvex friction law

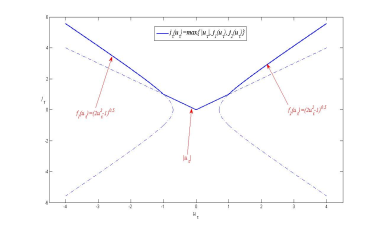


$$j(\xi) = \max\{f_1(\xi),\ f_2(\xi)\}$$

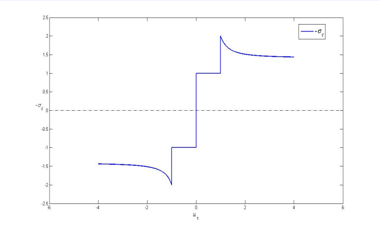


Clarke subgradient ∂j

Example: nonconvex friction law

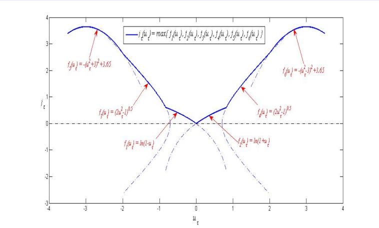


$$j(\xi) = \max\{a\|\xi\|,\ f_1(\xi),\ f_2(\xi)\}$$

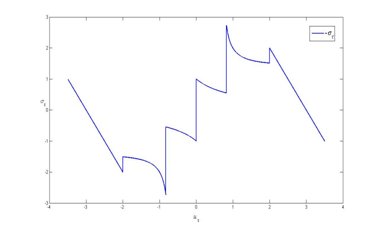


Clarke subgradient ∂j

Example: zig-zag friction law

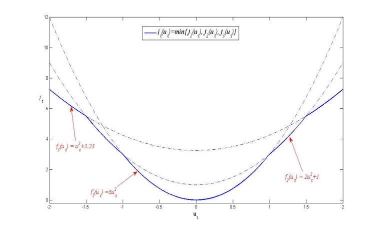


$$j(\xi) = \max\{f_1(\xi), f_2(\xi), f_3(\xi), f_1'(\xi), f_2'(\xi), f_3'(\xi)\}$$

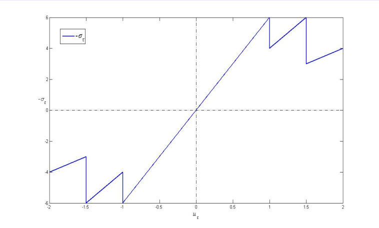


Clarke subgradient ∂j

Example: zig-zag friction law



$$j(\xi) = \min\{f_1(\xi),\ f_2(\xi),\ f_3(\xi)\}$$



Clarke subgradient ∂j

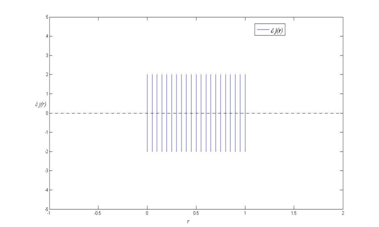
Example: infinite number of jumps

Let I be an open subset of the real line $\mathbb{R}$ and let M be a measurable subset of I such that for every open and nonempty subset I of I, $\mathrm{meas}(I \cap M) > 0$ and $\mathrm{meas}(I \cap (I \setminus M)) > 0$.
Let

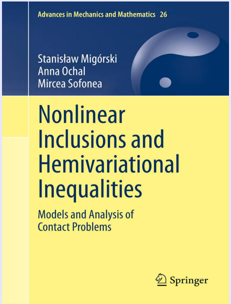
$$g(s) = \begin{cases} b_1 & \text{if } s \in M \\ -b_2 & \text{if } s \notin M \end{cases}$$

and $j(r) = \int_0^r g(\theta)\, d\theta$. Then the nonconvex potential j is locally Lipschitz and

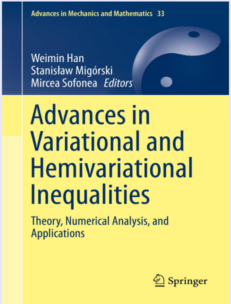
$$\partial j(r) = [-b_2, b_1] \ \text{ for every } \ r \in I.$$



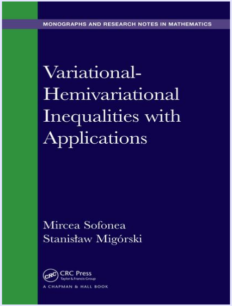
Monographs



2013



2015



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Thank you very much for your attention!