Convolutions of radial, exponential densities

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about joint work with Kamil Kaleta [1]

Plan of the presentation

Random walks

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$$S_n = \sum_{i=1}^{n} X_i \ , \qquad S_n \sim f^{n*} = \int_{\mathbb{R}^d} f(x-y) f^{(n-1)*}(y)dy.$$

Compound Possion measure

Let N be a variable with Poisson distribution,

independent from $\{X_i\}_{i\in\mathbb{N}}$ which is i.i.d..

We define

$$Y = \sum_{i=1}^{N} X_i.$$

Measure of such variable is,

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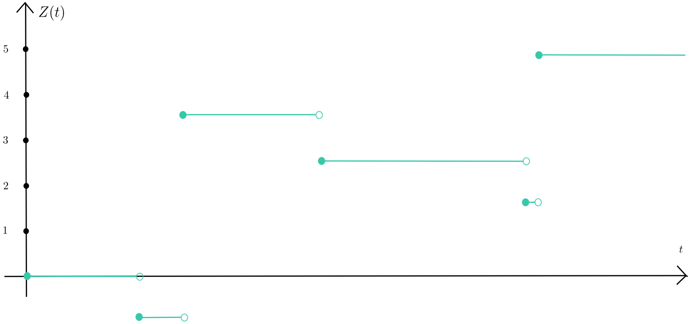
The absolute continuous part of that measure we denote by p_λ.

Similarly, we define Compound Poisson Measure,

$$Y(t) = \sum_{i=1}^{N(t)} X_i.$$

Typical trajectory



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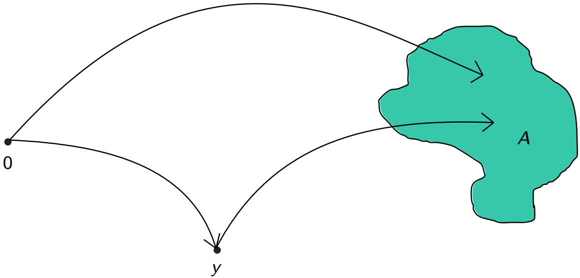
- Schrödinger semigroup theory [7, Kaleta, Lőrinczi];
- Convolution equivalence theory [6, Kaleta, Sztonyk] [5, Kaleta, Ponikowski].

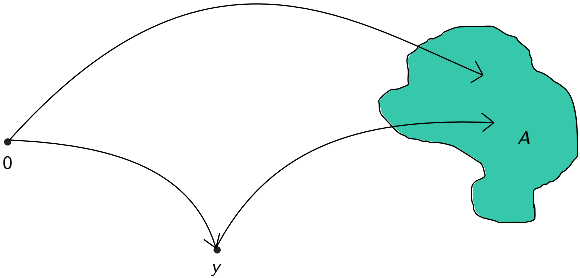
Convolution equivalence class

$$f^{2*}(x) \leqslant Cf(x)$$



Convolution equivalence class

$$f^{2*}(x) \leqslant Cf(x) \qquad f^{n*}(x) \leqslant C^{n-1}f(x)$$



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- what behaviour has p_λ?

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- If $\gamma \in \left[0, \frac{d+1}{2}\right]$ then $\lim_{|x|\to\infty} \frac{f^{2\star}(x)}{f(x)} = \infty.$

General framework

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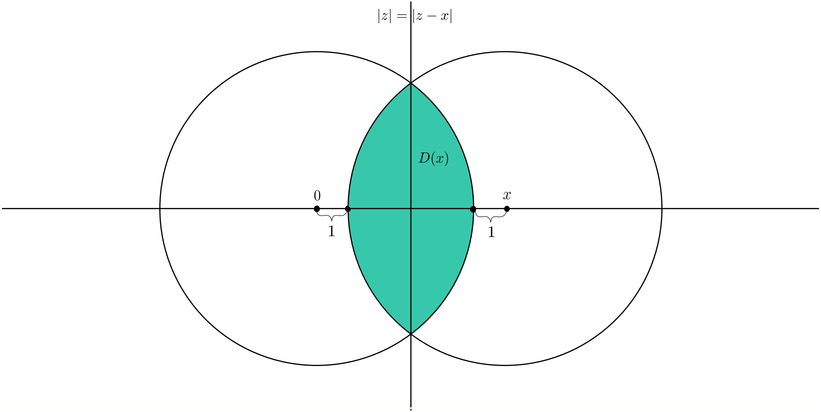
4. there exists a constant $C_2 \geqslant 1$, such $f(x) \leqslant C_2 f(2x)$ for $|x| \leqslant 1$.


$|z| = |z - x|$
$D(x)$
0
x
1
1

Functions h_n

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and by induction

$$h_{n+1}(x) := \frac{\int_{D(x)} f(x-y)f(y)h_n(y)dy}{f(x)}, \quad x \in \mathbb{R}^d, \quad n \geqslant 2.$$

Theorem

For $n \in \mathbb{N}$ and $|x| \geqslant 1$,

$$f^{n\star}(x) \asymp \left(\sum_{i=1}^{n} \binom{n}{i} C^{n-i} h_i(x) \right) f(x).$$

The constant C is different in both estimates.

Corollary

(a) *If there exists a constant $C > 0$, such $h_2(x) < C$, for every $x \in \mathbb{R}^d$, then*

$$f^{n\star}(x) \asymp nC^{n-1} f(x) \quad |x| \geqslant 1,\ n \in \mathbb{N}.$$

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$$\frac{f^{n\star}(x)}{f(x)} \xrightarrow{|x|\to\infty} \infty, \qquad \frac{f^{n\star}(x)}{f^{m\star}(x)} \xrightarrow{|x|\to\infty} 0 \text{ for } m > n.$$

Exponential densities

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$$H_{n+1}(r) := \frac{1}{g(r) r^{\frac{d-1}{2}}} \int_1^{r-1} g(r-\rho)(r-\rho)^{\frac{d-1}{2}} g(\rho) \rho^{\frac{d-1}{2}} H_n(\rho) d\rho.$$

Let $d \geqslant 2$, there exists constant $M > 0$ such

$$h_n(x) \leqslant M^{n-1} H_n(|x|), \quad x \in \mathbb{R}^d, \; n \geqslant 1,$$

Let's come back to the example of a function

$$f(x) = e^{-m|x|}|x|^{-\gamma}$$

where $m > 0$, $\gamma \in [0, \frac{d+1}{2})$.

Estimates of convolutions

Let $|x| \geqslant 1$, $n \in \mathbb{N}$. Let's denote $\rho_1 = d - \gamma$ and $\rho_2 = \frac{d+1}{2} - \gamma$. Then there exist constants D_1, D_2 such

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$$D_1^{n-1} \frac{\Gamma(\rho_1)^n}{\Gamma(\rho_1 n)} \leqslant \frac{f^{n\star}(x)}{f(x)|x|^{\left(\frac{d+1}{2}-\gamma\right)(n-1)}} \leqslant D_2^{n-1} \frac{\Gamma(\rho_2)^n}{\Gamma(\rho_2 n)} + O\left(\frac{1}{|x|^{\frac{d+1}{2}-\gamma}}\right).$$

Estimates of densities of compound Poisson measure

We have

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Because of that we have

$$\begin{aligned} p_\lambda(x) &= e^{-\lambda\|f\|_1} \sum_{n=1}^{\infty} \frac{\lambda^n f^{n\star}(x)}{n!} \\ &\asymp e^{-\lambda\|f\|_1} \sum_{n=1}^{\infty} \frac{\lambda^n |x|^{\left(\frac{d+1}{2}-\gamma\right)(n-1)} D^{n-1} \Gamma(\rho_i)^n}{\Gamma(\rho_i n) n!}. \end{aligned}$$

Generalized Bessel function:

$$\phi(\rho, \beta; t) := \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\rho n + \beta) n!}, \quad \rho > 0, \quad \beta \geqslant 0, \quad t > 0.$$

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It's asymptotic is described in [9, Wright].

Fact

There exist $D_1, D_2 > 0$ (depended on ρ i β) such, as

$$D_1 \leqslant \frac{\phi(\rho,\beta;t)}{t^{\frac{1-2\beta}{2\rho+2}} \exp\left((1+1/\rho)(\rho t)^{\frac{1}{\rho+1}}\right)} \leqslant D_2, \quad t \geqslant 1.$$

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and

$$\frac{p_\lambda(x)}{e^{-\lambda\|f\|_1} e^{-m|x|} |x|^{-\frac{d+1}{2}}} \leqslant e^{M_2\lambda} \phi(\rho_2, 0; \kappa_2 \lambda |x|^{\frac{d+1}{2}-\gamma}).$$

If $\lambda|x|^{\frac{d+1}{2}-\gamma} \geqslant 1$, then there exist constants E_1, E_2, E_3 and E_4 such

$$\frac{p_\lambda(x)}{e^{-\lambda\|f\|_1} e^{-m|x|} |x|^{-\frac{d+1}{2}}} \geqslant E_1 \exp\left(E_2 \left(\lambda |x|^{\frac{d+1}{2}-\gamma}\right)^{\frac{1}{\rho_1+1}} \right)$$

and

$$\frac{p_\lambda(x)}{e^{-\lambda\|f\|_1} e^{-m|x|} |x|^{-\frac{d+1}{2}}} \leqslant E_3 e^{\lambda M} \exp\left(E_4 \left(\lambda |x|^{\frac{d+1}{2}-\gamma}\right)^{\frac{1}{\rho_2+1}} \right).$$

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Thank you for your attention!