Decay of harmonic functions for discrete time Feynman-Kac operators with confining potentials

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- X - discrete, countably infinite state space.
- $P : X \times X \to [0,1]$ - a (sub-)probability kernel, i.e. $\sum_{y \in X} P(x,y) \leqslant 1$ for all $x \in X$,
- Equivalently, there exists a time-homogeneous Markov chain on X $\{Y_n : n \in \mathbb{N}_0\}$ such that for all $x, y \in X$

$$\mathbb{P}(Y_{n+1} = y | Y_n = x) = P(x,y).$$

Discrete Feynman–Kac semigroups

Let $V : X \to (0, \infty)$ be a function such that $\inf_{x \in X} V(x) > 0$. Such function is called a *potential*.
We define a semigroup of operators $\{\mathcal{U}_n : n \in \mathbb{N}_0\}$:

$$\mathcal{U}_0 f = f, \quad \mathcal{U}_n f(x) = \mathbb{E}^x \left[\prod_{k=0}^{n-1} \frac{1}{V(Y_k)} f(Y_n) \right], \quad n \geqslant 1$$

for all admissible functions f. Observe that $\mathcal{U}_n = \mathcal{U}^n$, where

$$\mathcal{U} f(x) = \frac{1}{V(x)} \sum_{y \in X} P(x, y) f(y), \quad x \in X.$$

The operator $\mathcal{U} - I$ is called *the discrete Feynman–Kac operator*.

Relations between F-K and Schrödinger operators

Consider a discrete Schrödinger operator

$$H f(x) = \sum_{y \in X} P(x,y)(f(x) - f(y)) + V(x)f(x),$$

where $\inf_{x \in X}(V(x) + \sum_{y \in X} P(x,y)) > 0$. Then we have

$$\frac{1}{V(x) + \sum_{y \in X} P(x,y)} H f(x) = (I - \mathcal{U})f(x),$$

where $\mathcal{U}$ is defined with a *shifted potential*

$$V_*(x) = V(x) + \sum_{y \in X} P(x,y).$$

This implies H and $(I - \mathcal{U})$ share many analytic properties.

Our goal is to obtain estimates for $(\mathcal{U} - I)$-harmonic functions. We will focus on two important cases:

- Markov chains with the direct step property
- nearest-neighbour random walks

DSP with confining potentials

To find satisfactory estimates, we assume the following:

(A) We have $P(x, y) > 0$ for all $x, y \in X$, and there exists $C_* > 0$ such that

$$\sum_{z \in X} P(x, z)P(z, y) \leqslant C_* \, P(x, y).$$

This property is called the *direct step property* (DSP).

(B) For all $M > 0$, there exists a finite set $B_M \subset X$ such that $V(x) \geqslant M$ for $x \in B_M^c$. A potential with this property is called a *confining potential*.

DSP covers many cases

In this section, we assume that (X, d) is a metric space.

Theorem

Let $P(x,y)$ be a (sub-)probability kernel such that

$$P(x,y) \asymp J(d(x,y)), \quad x,y \in X, \tag{1}$$

for a non-increasing function $J : [0,\infty) \to (0,\infty)$ which satisfies the following doubling condition: there exists a constant $C > 0$ such that

$$J(r) \leqslant CJ(2r), \quad \text{for all } r > 0. \tag{2}$$

Then the kernel $P(x,y)$ satisfies assumption (A).

DSP covers many cases

Theorem

Let $P(x,y)$ be a (sub-)probability kernel such that

$$P(x,y) \asymp J(d(x,y))K(d(x,y)), \quad x,y \in X, \tag{3}$$

where $J, K : [0,\infty) \to (0,\infty)$ are non-increasing functions such that J satisfies (2) and K is such that

$$K(r)K(s) \leqslant \widetilde{C} K(r+s), \quad r,s > 0. \tag{4}$$

Then the kernel $P(x,y)$ satisfies assumption (A).

DSP can also be inherited from a subordinator.

- $\{Z_n : n \geqslant 0\}$ - time-homogeneous Markov chain with values in X,
- $\{\tau_n : n \geqslant 0\}$ - arbitrary increasing random walk starting at 0 with values in $\mathbb{N}_0$, which is independent of $\{Z_n : n \geqslant 0\}$ (by saying that it is a random walk we mean that $\tau_{n+1} - \tau_n$, $n = 0, 1, 2. \ldots$ are i.i.d. random variables)

The *subordinate Markov chain* $\{Y_n : n \geqslant 0\}$ is then defined as

$$Y_n := Z_{\tau_n}, \quad n = 0, 1, 2, \ldots$$

Lemma

Suppose $\{\tau_n\}$ *has the DSP, that is*

$$\mathbb{P}(\tau_2 = n) \leqslant C_* \mathbb{P}(\tau_1 = n), \quad n = 2, 3, \ldots \tag{5}$$

for some constant $C_* > 0$. *Then* $\{Y_n\}$ *inherits the DSP with the same constant.*

Proof

Since $\mathbb{P}(\tau_2 = 1) = 0$, *for any* $x, y \in X$ *we have*

$$\begin{aligned}\mathbb{P}(Y_2 = y \mid Y_0 = x) &= \sum_{k=1}^{\infty} \mathbb{P}(Z_k = y \mid Z_0 = x)\mathbb{P}(\tau_2 = k) \\ &\leqslant C_* \sum_{k=1}^{\infty} \mathbb{P}(Z_k = y \mid Z_0 = x)\mathbb{P}(\tau_1 = k) \\ &= C_* \, \mathbb{P}(Y_1 = y \mid Y_0 = x),\end{aligned}$$

as desired.

Sufficient condition for assumption (A)

Theorem

If $\{Z_n : n \geqslant 0\}$ *is irreducible,* $\{\tau_n : n \geqslant 0\}$ *is such that* (5) *holds and there exists* $n_0 \in \mathbb{N}$ *such that*

$$\mathbb{P}(\tau_1 = n) > 0, \quad n \geqslant n_0, \tag{6}$$

then the subordinate chain $\{Y_n : n \geqslant 0\}$ *satisfies assumption (A).*

Estimates for harmonic functions

For every finite set $B \subset X$ we define

$$\underline{K}_B := \inf \left\{ \frac{P(x,y)}{P(x,z)} : x \in X; y, z \in B \right\},$$
$$\overline{K}_B := \sup \left\{ \frac{P(x,y)}{P(x,z)} : x \in X; y, z \in B \right\}.$$

It is easy to check that $0 < \underline{K}_B \leqslant \overline{K}_B < \infty$. We fix a finite set $B_0 \subset X$ such that

$$C_1 := \sup \left\{ \frac{1}{V(x)} : x \in B_0^c \right\} < 1 \wedge \frac{1}{C_*}.$$

The existence of such a set is secured by assumption (B). Note that B_0 depends on V and P.

Theorem

Under assumptions (A) and (B), there exists a constant $C_2 > 0$ *such that for any finite set* $B \subset X$ *with* $B \supseteq B_0$*, and for any non-negative bounded function* f *which is subharmonic in* B^c *we have*

$$f(x) \leqslant C_2 \frac{1}{V(x)} \sum_{y \in B} P(x,y) f(y), \quad x \in B^c.$$

In particular,

$$f(x) \leqslant C_2 \overline{K}_B \frac{P(x,x_0)}{V(x)} \sum_{y \in B} f(y), \quad x \in B^c, \; x_0 \in B.$$

The constant C_2 *depends neither on* f*,* V*, nor on the set* B*.*

Theorem

For any $D \subset X$, any non-negative function f which is superharmonic in D, and for any finite set $B \subset X$ we have

$$f(x) \geqslant \frac{1}{V(x)} \sum_{y \in B} P(x,y) f(y), \quad x \in D.$$

In particular, under assumption (A),

$$f(x) \geqslant \underline{K}_B \frac{P(x,x_0)}{V(x)} \sum_{y \in B} f(y), \quad x \in D, \; x_0 \in B.$$

Theorem (Cygan, Kaleta, MŚ)

*Under assumptions **(A)** and **(B)**, for any finite set $B \subset X$ with $B \supseteq B_0$, for any set $D \subset X$, and for any non-negative, non-zero and bounded function f which is harmonic in D and such that $f(x) = 0$ for $x \in D^c \cap B^c$ we have*

$$\underline{K}_B \leqslant \frac{f(x)}{\frac{P(x,x_0)}{V(x)} \sum_{y \in B} f(y)} \leqslant C_2 \overline{K}_B, \quad x \in D \cap B^c, \ x_0 \in B,$$

where C_2 is independent of f, V and B.
*In particular, the **uniform Boundary Harnack Inequality at infinity** holds: if f and g are two such non-zero harmonic functions, then*

$$\left(\frac{\underline{K}_B}{C_2 \overline{K}_B} \right)^2 \leqslant \frac{f(x)g(y)}{g(x)f(y)} \leqslant \left(\frac{C_2 \overline{K}_B}{\underline{K}_B} \right)^2, \quad x, y \in D \cap B^c.$$

Nearest-neighbour random walks

First we impose a graph structure on X.

- Graph $G = (X, E)$ is defined by specifying a set of *edges* $E \subset \{\{x, y\} : x, y \in X\}$. Two vertices $x, y \in X$ are connected by an edge (are neighbours) in G iff $\{x, y\} \in E$ (we have $\{x, y\} = \{y, x\}$). Notation: $x \sim y$.
- G is said to be *connected*, if every two different vertices x and y are connected by a path in G.
- G is said to be of *finite geometry*, if every vertex has finitely many neighbours.

Throughout this section we assume that

(C) G is a connected graph of finite geometry.

This allows us to impose a *geodesic metric* d on G, where $d(x, y)$ is the length of the shortest path between x and y.

We consider a (sub-)probability kernel P such that

$$P(x, y) > 0 \iff x \sim y. \tag{7}$$

We restrict our attention to potentials with the following property:

(D) There exist $x_0 \in X$ and an increasing profile function $W : \mathbb{N}_0 \to (0, \infty)$ such that $V(x) = W(d(x_0, x))$, for any $x \in X$.

Additionally, we use $B_r(x_0)$ to denote an open ball (with respect to d) with radius r and center x_0.

Theorem

Let assumptions (C) and (D) hold with a fixed $x_0 \in X$ and a profile function W. Let $\mathcal{U} - I$ be the Feynman–Kac operator corresponding to the kernel $P(x, y)$ satisfying (7). Then for any $r \in \mathbb{N}$ and for any non-negative and bounded function f which is $(\mathcal{U} - I)$-subharmonic in $B_r(x_0)^c$ we have

$$f(x) \leqslant \|f\|_\infty \prod_{i=r}^{d(x,x_0)} \frac{1}{W(i)}, \quad x \in B_r(x_0)^c.$$

To obtain the lower bound for $(\mathcal{U} - I)$-superharmonic functions we consider connected and geodesically convex subsets of X.

The set $D \subset X$ is called *geodesically convex* in a graph $G = (X, E)$ if D contains each vertex on any geodesic path connecting vertices in D.

We also need an additional regularity assumption on the kernel $P(x, y)$:

$$M := \inf \{P(x, y) : x, y \in X,\ x \sim y\} > 0. \tag{8}$$

Theorem

Let assumptions (C) and (D) hold with some $x_0 \in X$ and a profile function W. Let $\mathcal{U} - I$ be the Feynman–Kac operator corresponding to the kernel $P(x, y)$ satisfying (7) and (8). Then, for any connected geodesically convex set $D \subset X$, for any non-negative function f which is $(\mathcal{U} - I)$-superharmonic in D, for any $x \in D$, and for any $x_r \in D$ which lies on the geodesic path connecting x with x_0 and is such that $d(x_r, x_0) = r < d(x, x_0)$, we have

$$f(x) \geqslant f(x_r) \prod_{i=r+1}^{d(x,x_0)} \frac{M}{W(i)}.$$

Decay of solutions to equations of graph Laplacians

Consider a kernel $b : X \times X \to [0, \infty)$ such that

(i) $b(x, y) = b(y, x)$, for every $x, y \in X$;

(ii) $\sum_{y \in X} b(x, y) > 0$, for every $x \in X$, and $\sup_{x \in X} \sum_{y \in X} b(x, y) < \infty$.

Let $m : X \to (0, \infty)$ be a (strictly positive) measure on X. We additionally consider a function $V : X \to \mathbb{R}$ such that $\inf_{x \in X} V(x) > -\infty$. The *graph Laplacian* H is defined by

$$Hf(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)(f(x) - f(y)) + V(x)f(x),$$

for all functions
$f \in F := \{f : X \to \mathbb{R} : \sum_y b(x, y)|f(y)| < \infty, \text{ for every } x \in X\}$.
The triple (X, b, V) can be seen as a weighted graph over X (two points $x, y \in X$ form an edge if and only if $b(x, y) > 0$).

We set

$$b(x) = \sum_{y \in X} b(x,y), \qquad b^* := \sup_{x \in X} b(x), \qquad P(x,y) = \frac{b(x,y)}{b^*}, \tag{9}$$

and

$$\widetilde{V}(x) = \begin{cases} \frac{m(x)V(x)+b(x)}{b^*}, & x \in A^c, \\ 1, & x \in A, \end{cases} \tag{10}$$

where $A = \{x \in X : m(x)V(x) + b(x) \leqslant b^*\}$.

We further assume that the operator $\mathcal{U}$ is defined with a sub-probability kernel $P(x,y)$ and the potential $\widetilde{V}(x)$ defined at (9) and (10).

Theorem

For every $f \in F$ *and* $x \in A^c$ *we have*

$$Hf(x) = -\left(V(x) + \frac{b(x)}{m(x)}\right)\ (\mathcal{U} - I)f(x).$$

In particular, if $D \subset A^c$ *and* $f \in F$*, then*

$$Hf(x) \geqslant 0, \quad x \in D \qquad \Longleftrightarrow \qquad (\mathcal{U} - I)f(x) \leqslant 0, \quad x \in D.$$

Corollary (DSP case, upper bound)

Suppose that $b(x,y)$, $m(x)$ *and* $V(x)$ *are as above. Assume that* V *satisfies (B),* $\inf_{x\in X} m(x) > 0$ *and that*

$$b(x,y) > 0, \qquad \sup_{x,y\in X} \sum_{z\in X} \frac{b(x,z)b(z,y)}{b(x,y)} < \infty. \tag{11}$$

Let $D \subset X$ *and let* f *be a bounded solution to the equation* $Hf(x) = 0$, $x \in D$. *Then there exists a finite set* $B_0 \subset X$ *(independent of* m, D *and* f*) with* $B_0 \supseteq A$ *such that for any finite set* $B \subset X$ *with* $B \supseteq B_0$ *there exists a constant* $C > 0$ *(independent of* V, m, D *and* f*) such that*

$$|f(x)| \leqslant C \frac{b(x,x_0)}{m(x)V(x) + b(x)} \sum_{y\in B} |f(y)|, \quad x \in D \cap B^c,\ x_0 \in B,$$

whenever $f(x) = 0$ *for* $x \in D^c \cap B^c$.

Corollary (DSP case, lower bound)

If, in addition, f is non-negative, then for any finite set $B \subset X$ with $B \supseteq B_0$ there exists a constant $\widetilde{C} > 0$ (independent of V, m, D and f) such that

$$f(x) \geqslant \widetilde{C} \frac{b(x, x_0)}{m(x)V(x) + b(x)} \sum_{y \in B} f(y), \quad x \in D \cap B^c, \ x_0 \in B.$$

Proof. Observe that when V is a confining potential, then $\widetilde{V}$ is confining as well. Realize that f is $(\mathcal{U} - I)$-harmonic in $D \cap A^c$. To justify the upper bound, it is enough to observe that $|f|$ is $(\mathcal{U} - I)$-subharmonic in B^c and apply respective theorem. The corresponding lower bound is obtained directly from the proposition in the DSP case.

Eigenfunctions of discrete F-K operators

Suppose we are given a positive measure μ on X such that

$$\text{(i)} \ \sup_{y \in X} \frac{\sum_{x \in X} \mu(x) P(x,y)}{\mu(y)} < \infty, \quad \text{(ii)} \ \sup_{x,y \in X} \frac{P(x,y)}{\mu(y)} < \infty.$$

Under condition (i), the operator $\mathcal{U}$ is bounded in $\ell^p(X,\mu)$, for any $1 \leqslant p < \infty$. Condition (ii) implies that the operator $\mathcal{U} : \ell^p(X,\mu) \to \ell^\infty(X,\mu)$ is bounded for every $1 \leqslant p < \infty$.

Lemma

Under assumption (B), the operator $\mathcal{U}$ is compact in $\ell^2(X,\mu)$.

We deduce that the spectrum of the operator $\mathcal{U}$ (excluding zero) consists solely of eigenvalues. Moreover, by Jentzsch theorem, the spectral radius of $\mathcal{U}$ is an eigenvalue, which we denote by $\lambda_0 > 0$, and the corresponding eigenfunction ψ_0 is strictly positive on X.

Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$ be an eigenvalue of the operator $\mathcal{U}$ and let $\psi \in \ell^2(X, \mu)$ be the corresponding eigenfunction, i.e. $\mathcal{U}\psi = \lambda\psi$. We then have $|\lambda||\psi| = |\mathcal{U}\psi| \leqslant \mathcal{U}|\psi|$, which implies $|\psi| \leqslant \mathcal{U}^\lambda|\psi|$, where

$$\mathcal{U}^\lambda f(x) = \frac{1}{V_\lambda(x)} \sum_{y \in X} P(x, y)|\psi(y)|, \quad \text{with } V_\lambda := |\lambda| V.$$

In particular, $(\mathcal{U}^\lambda - I)|\psi|(x) \geqslant 0$, $x \in X$, i.e. the non-negative function $\varphi := |\psi|$ is $(\mathcal{U}^\lambda - I)$-subharmonic in X. We show similarly that the positive function ψ_0 is $(\mathcal{U}^\lambda - I)$-harmonic.

After this preparation we can apply our results to obtain an upper bound for $|\psi|$ outside of a finite set in the DSP and the nearest-neighbour case, respectively.

We can also find the matching lower bound for the positive eigenfunction ψ_0 in these two cases.

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Thank you!