Monotonicity and discretization of integral operators

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Basic definitions

Let $(X, \|\cdot\|)$ is a real Banach space.
A nonempty closed and convex subset $X_+ \subset X$ is called order cone, if $\mathbb{R}_+ X_+ \subset X_+$ and $X_+ \cap (-X_+) = \{0\}$ hold.
Let us assume $X_+ \neq \{0\}$ throughout and for elements $x, \bar{x} \in X$ we write

$$x \leq \bar{x} \;:\Leftrightarrow\; \bar{x} - x \in X_+,$$

$$x < \bar{x} \;:\Leftrightarrow\; \bar{x} - x \in X_+ \setminus \{0\},$$

$$x \ll \bar{x} \;:\Leftrightarrow\; \bar{x} - x \in X_+^o,$$

where the latter relation requires $X_+^o \neq \emptyset$ (X_+^o is the interior of X_+) and one speaks of a solid cone X_+.

Basic definitions

Let X is a real Banach space, $X_+ \subset X$ a cone, $U \subseteq X$.
A mapping $F : U \to X$ is called

- monotone, if
$$x < \bar{x} \Rightarrow F(x) \leq F(\bar{x}),$$
- strictly monotone, if
$$x < \bar{x} \Rightarrow F(x) < F(\bar{x}),$$
- strongly monotone, if
$$x < \bar{x} \Rightarrow F(x) \ll F(\bar{x}),$$

for all $x, \bar{x} \in U$.
In particular, a linear mapping $T : X \to X$ is

- monotone (then called positive), if $T(X_+ \setminus \{0\}) \subseteq X_+$,
- strictly monotone (then called strictly positive), if $T(X_+ \setminus \{0\}) \subseteq X_+ \setminus \{0\}$,
- strongly monotone (then called strongly positive), if $T(X_+ \setminus \{0\}) \subseteq X_+^\circ$.

Basic definitions

We equip a compact metric space Ω with a σ-algebra $\mathfrak{A}$ (containing the Borel sets) and a measure μ such that $(\Omega, \mathfrak{A}, \mu)$ is a measure space satisfying $\mu(\Omega) < \infty$. The set $C(\Omega)^d$ of all continuous functions $u : \Omega \to \mathbb{R}^d$ is a real Banach space with norm $\|u\|_\infty := \max_{x\in\Omega} |u(x)|$. Moreover,

$$C(\Omega)^d_+ := \left\{u \in C(\Omega)^d : \, u(x) \in Y_+ \text{ for all } x \in \Omega\right\}$$

abbreviates the set of continuous functions having values in the cone $Y_+ \subset \mathbb{R}^d$.

Lemma

The set $C(\Omega)^d_+$ is a cone, which is solid, provided Y_+ is.

Basic definitions

Having identified $C(\Omega)^d_+$ as (solid) cone, we introduce the relations

$$\begin{aligned} u \preceq \bar{u} \quad &:\Leftrightarrow \quad \bar{u} - u \in C(\Omega)^d_+, \\ u \prec \bar{u} \quad &:\Leftrightarrow \quad \bar{u} - u \in C(\Omega)^d_+ \setminus \{0\}, \\ u \prec\!\!\prec \bar{u} \quad &:\Leftrightarrow \quad \bar{u} - u \in (C(\Omega)^d_+)^\circ \quad \text{for all } u, \bar{u} \in C(\Omega)^d. \end{aligned}$$

Lemma

The following holds for $u, \bar{u} \in C(\Omega)^d$:

1. $u \preceq \bar{u} \Leftrightarrow u(x) \leq \bar{u}(x)$ *for all* $x \in \Omega \Leftrightarrow \langle u(x), y' \rangle \leq \langle \bar{u}(x), y' \rangle$ *for all* $x \in \Omega$ *and* $y' \in Y'_+$.
2. $u \prec \bar{u} \Leftrightarrow u(x) \leq \bar{u}(x)$ *for all* $x \in \Omega$ *and* $u(x_0) < \bar{u}(x_0)$ *for some* $x_0 \in \Omega$.
3. *If* Y_+ *is solid, then one has*
   $u \prec\!\!\prec \bar{u} \Leftrightarrow u(x) \ll \bar{u}(x)$ *for all* $x \in \Omega \Leftrightarrow \langle u(x), y' \rangle < \langle \bar{u}(x), y' \rangle$ *for all* $x \in \Omega$, $y' \in Y'_+ \setminus \{0\}$.

Fredholm integral operators

Now we want to provide sufficient conditions for Fredholm integral operators

$$\mathcal{K}u := \int_\Omega K(\cdot, y)u(y)\,\mathrm{d}\mu(y)$$

to preserve order relations. In essence, we demonstrate that monotonicity(positivity) properties of the kernel functions carry over to the integral operators.
We consider Fredholm operators under assumption for $K : \Omega \times \Omega \to L(\mathbb{R}^d)$:

Hypothesis

(L) $K(x, \cdot) : \Omega \to L(\mathbb{R}^d)$ *is* μ*-measurable for all* $x \in \Omega$ *with*

$$\sup_{x\in\Omega} \int_\Omega |K(x, y)|\,\mathrm{d}\mu(y) < \infty \text{ and}$$

$$\lim_{x\to x_0} \int_\Omega |K(x, y) - K(x_0, y)|\,\mathrm{d}\mu(y) = 0 \text{ for all } x_0 \in \Omega,$$

which yield that $\mathcal{K} \in L(C(\Omega)^d)$.

Fredholm integral operators

Theorem (positivity of $\mathcal{K}$ on $C(\Omega)^d$)

Let Hypothesis (L) hold. If $K(x,y)$ is Y_+-positive for all $x \in \Omega$ and μ-a.a. $y \in \Omega$, then a Fredholm operator $\mathcal{K} \in L(C(\Omega)^d)$ is $C(\Omega)^d_+$-positive.

Furthermore we denote $T \in L(X)$ as X_+-*injective*, provided its kernel satisfies $N(T) \cap X_+ = \{0\}$.

Theorem (strictly positivity of $\mathcal{K}$ on $C(\Omega)^d$)

Let Hypothesis (L) hold and $K(x,y)$ is Y_+-positive for all $x \in \Omega$ and μ-a.a. $y \in \Omega$. If nonempty, open subsets of Ω have positive measure, there exists a $\bar{x} \in \Omega$ so that $K(\bar{x},\cdot)$ is continuous and $K(\bar{x},y)$ is Y_+-injective for μ-a.a. $y \in \Omega$, then $\mathcal{K}$ strictly $C(\Omega)^d_+$-positive.

Theorem (strongly positivity of $\mathcal{K}$ on $C(\Omega)^d$)

Let Hypothesis (L) holds. If nonempty, open subsets of Ω have positive measure, Y_+ is solid and $K(x,y)$ is strongly Y_+-positive for all $x \in \Omega$ and μ-almost all $y \in \Omega$, then $\mathcal{K}$ is strongly $C(\Omega)^d_+$-positive.

Nyström methods

We suppose $\Omega \subset \mathbb{R}^\kappa$ is compact with positive Lebesgue measure $\lambda_\kappa(\Omega) > 0$. Given a continuous function $u : \Omega \to \mathbb{R}^d$, consider the representation

$$\int_\Omega u(y)\,\mathrm{d}y = \sum_{j=0}^{q_n} w_j u(\eta_j) + E_n(u) \tag{Q_n}$$

with a sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$, nodes from a finite set $\Omega_n := \{\eta_0, \ldots, \eta_{q_n}\} \subseteq \Omega$, weights $w_j \in \mathbb{R}$ such that the remainder (error term) satisfies $\lim_{n\to\infty} E_n(u) = 0$. Such schemes are called *convergent*. We say that an integration rule (Q_n) fulfills the net condition, if

$$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N} : \; \Omega \subseteq \bigcup_{j=0}^{q_n} B_\varepsilon(\eta_j) \quad \text{for all } n \geq n_0(\varepsilon).$$

Nyström methods

A natural way to evaluate the Lebesgue integral in the Fredholm operator is to apply a rule (Q_n).

Hypothesis

Assume that a kernel $K : \Omega \times \Omega \to L(\mathbb{R}^d)$ fulfills:

(NL) *$K(\cdot, y) : \Omega \to L(\mathbb{R}^d)$ is continuous for all $y \in \Omega$.*

This leads to the (spatially) *discrete Fredholm integral operator*

$$\mathcal{K}^n u := \sum_{j=0}^{q_n} w_j K(\cdot, \eta_j) u(\eta_j).$$

There are two natural choices for the domain of $\mathcal{K}^n$, namely a spatially continuous one $C(\Omega)^d$ and the spatially discrete function space

$$C(\Omega_n)^d = \left\{ u : \Omega_n \to \mathbb{R}^d \right\};$$

both are equipped with the max-norm. In each case, (NL) suffices to obtain that $\mathcal{K}^n$ is well-defined.

$$\mathcal{K}^n u := \sum_{j=0}^{q_n} w_j K(\cdot, \eta_j) u(\eta_j)$$

Remark ($\mathcal{K}^n$ on the domain $C(\Omega_n)^d$)

Suppose that (Q_n) has nonnegative weights. Then previous Theorem (positivity of $\mathcal{K}$ on $C(\Omega)^d$) applies in the special case $\mu = \mu_n$ and guarantees that positivity of $\mathcal{K}^n \in L(C(\Omega_n)^d)$ or $\mathcal{K}^n \in L(C(\Omega_n)^d, C(\Omega)^d)$ holds literally with the assumption "μ-a.a. $y \in \Omega$" replaced by "all $y \in \Omega_n$".

Remark ($\mathcal{K}^n$ on the domain $C(\Omega)^d$)

On the domain $C(\Omega)^d$ one cannot expect $\mathcal{K}^n$ to be strictly or strongly positive. This is due to the fact that $C(\Omega)^d_+ \setminus \{0\}$ contains functions u vanishing except from being positive on arbitrarily small domains disjoint from Ω_n. Hence, they are not captured by the Nyström grid Ω_n, that is, $u|_{\Omega_n} = 0$ although $u \neq 0$. Consequently, one has $\mathcal{K}^n u = 0$.

Nyström methods

Theorem (positivity of $\mathcal{K}^n$ on $C(\Omega)^d$)

Let $K(\cdot, y) : \Omega \to L(\mathbb{R}^d)$ is continuous for all $y \in \Omega$ hold.

1. *If (Q_n), $n \in \mathbb{N}$, have nonnegative weights and $K(x, \eta)$ is Y_+-positive for all $x \in \Omega$, $\eta \in \Omega_n$, then $\mathcal{K}^n \in L(C(\Omega)^d)$ is $C(\Omega)^d_+$-positive.*
2. *If (Q_n), $n \in \mathbb{N}$, have positive weights, Y_+ is solid and $K(x, \eta)Y_+^\circ \subseteq Y_+^\circ$ for all $x \in \Omega$, $\eta \in \Omega_n$, then $\mathcal{K}^n(C(\Omega)^d_+)^\circ \subseteq (C(\Omega)^d_+)^\circ$.*

Nyström methods

Theorem (eventually positivity of $\mathcal{K}^n$ on $C(\Omega)^d$)

Let $K(\cdot, y) : \Omega \to L(\mathbb{R}^d)$ is continuous for all $y \in \Omega$, $\Omega = \overline{\Omega^\circ}$ and (Q_n) have eventually positive weights and the net condition hold, then for each $u \in C(\Omega)^d$ with $0 \prec u$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$:

1. *If $K(x, \eta)$ is Y_+-positive for all $x \in \Omega$, $\eta \in \Omega_n$ and $K(\bar{x}, \eta)$ is Y_+-injective for one $\bar{x} \in \Omega$ and all $\eta \in \Omega_n$, then $0 \prec \mathcal{K}^n u$.*
2. *If Y_+ is solid and $K(x, \eta)$ is strongly Y_+-positive for all $x \in \Omega$, $\eta \in \Omega_n$, then $0 \lll \mathcal{K}^n u$.*

Collocation method

In projection method we consider a linear projection $\Pi_n \in L(C(\Omega)^d, X_n^d)$, where X_n^d is a subspace of $C(\Omega)^d$ with $X_n = \mathrm{lin}\{\phi_1, \ldots, \phi_{d_n}\}$. The spatial discretizations of an integral operator $\mathcal{K} \in L(X(\Omega)^d)$ become

$$\mathcal{K}^n := \Pi_n \mathcal{K}.$$

One of projection method is a collocation method, in which pairwise different *collocation points* $x_1, \ldots, x_{d_n} \in \Omega$ satisfy the interpolation conditions and follow to collocation matrix

$$[\phi_i(x_j)]_{i,j=1}^{d_n}$$

which is denoted by P_n.

Collocation method

Theorem (positivity of Π_n on $C(\Omega)^d$)

If all the functions

$$\sigma_i : \Omega \to \mathbb{R}, \qquad \sigma_i(x) := \sum_{j=1}^{d_n} (P_n^{-1})_{ij} \phi_j(x) \quad \textit{for all } 1 \le i \le d_n$$

have nonnegative values, then the following hold:

1. Π_n *is* $C(\Omega)^d_+$*-positive.*
2. *If additionally* Y_+ *is solid and*

$$\forall x \in \Omega : \exists i_0 \in \{1, \dots, d_n\} : \sigma_{i_0}(x) > 0$$

holds, then $\Pi_n(C(\Omega)^d_+)^\circ \subset (C(\Omega)^d_+)^\circ$.

Example

$$P_n = [\phi_i(x_j)]_{i,j=1}^{d_n} \quad \sigma_i(x) := \sum_{j=1}^{d_n} (P_n^{-1})_{ij} \phi_j(x) \quad \text{for all } 1 \leq i \leq d_n$$

On the other hand, the applicability of Theorem (positivity of Π_n on $C(\Omega)^d$) is hindered by the following fact: Many bases $\phi_1, \dots, \phi_{d_n}$ (e.g. B-splines, Bernstein polynomials, etc.) consist of functions having nonnegative values yielding a nonnegative collocation matrix P_n. Thus, P_n^{-1} has nonnegative entries, if and only if P_n is a monomial matrix, i.e. every column/row contains exactly one positive element.
Positive projections have:

- piecewise linear collocation;
- polynomial interpolation;
- cubic spline.

Projection which is not positive: quadratic splines.

Collocation method

Corollary (positivity of $\mathcal{K}^n$ on $C(\Omega)^d$)

Let Hypothesis (L) hold and all the functions

$$\sigma_i : \Omega \to \mathbb{R}, \qquad \sigma_i(x) := \sum_{j=1}^{d_n} (P_n^{-1})_{ij} \phi_j(x) \quad \text{for all } 1 \leq i \leq d_n$$

have nonnegative values.

1. *If $\mathcal{K}$ is $C(\Omega)^d_+$-positive, then $\mathcal{K}^n = \Pi_n \mathcal{K} \in L(C(\Omega)^d, X_n)$ and $\mathcal{K}\Pi_n \in L(C(\Omega)^d)$ are $C(\Omega)^d_+$-positive.*
2. *If $\mathcal{K}$ is strongly $C(\Omega)^d_+$-positive, Y_+ is solid and*

   $$\forall x \in \Omega : \exists i_0 \in \{1, \dots, d_n\} : \sigma_{i_0}(x) > 0$$

   holds, then $\mathcal{K}^n = \Pi_n \mathcal{K} \in L(C(\Omega)^d, X_n)$ and $\mathcal{K}\Pi_n \in L(C(\Omega)^d)$ are strongly $C(\Omega)^d_+$-positive.

Urysohn integral operators

$$\mathcal{F} : U \to C(\Omega)^d, \qquad \mathcal{F}(u) := \int_\Omega f(\cdot, y, u(y))\,\mathrm{d}\mu(y), \qquad (F)$$

$$U := \left\{u \in C(\Omega)^d : \ u(x) \in Z \text{ for all } x \in \Omega\right\}.$$

Let $Z \subseteq \mathbb{R}^d$ be nonempty. Assume for a kernel function $f : \Omega^2 \times Z \to \mathbb{R}^d$:

Hypothesis

(U^0) *$f(x, \cdot, z) : \Omega \to \mathbb{R}^d$ is μ-measurable for all $x \in \Omega$, $z \in Z$, for every $r > 0$ there exists a μ-measurable function $\beta_r^0 : \Omega^2 \to \mathbb{R}_+$ satisfying*

$$\sup_{x\in\Omega} \int_\Omega \beta_r^0(x, y)\,\mathrm{d}\mu(y) < \infty,$$

such that for μ-a.a. $y \in \Omega$ it is $|f(x, y, z)| \le \beta_r^0(x, y)$ for all $x \in \Omega$, $z \in Z \cap \bar{B}_r(0)$ and $f(\cdot, y, \cdot) : \Omega \times Z \to L(\mathbb{R}^d)$ exists as continuous function for μ-a.a. $y \in \Omega$. Furthermore, for every $r > 0$ there exist a μ-measurable function $\gamma_r^0 : \Omega^3 \to \mathbb{R}_+$ satisfying

$$\lim_{x\to x_0} \int_\Omega \gamma_r^0(x, x_0, y)\,\mathrm{d}\mu(y) = 0 \quad \textit{for all } x_0 \in \Omega,$$

such that for μ-a.a. $y \in \Omega$ one has $|f(x, y, z) - f(\bar{x}, y, z)| \le \gamma_r^0(x, \bar{x}, y)$ for all $x, \bar{x} \in \Omega$, $z \in Z \cap \bar{B}_r(0)$.

Urysohn integral operators

Theorem (properties of $\mathcal{F}$)

Let Hypothesis (U^0) *hold.*

1. *If for all* $x \in \Omega$ *and* μ*-a.a.* $y \in \Omega$ *a kernel function* $f(x, y, \cdot) : Z \to \mathbb{R}^d$ *is* Y_+*-monotone, then an Urysohn operator* $\mathcal{F} : U \to C(\Omega)^d$ *is* $C(\Omega)^d_+$*-monotone.*
2. *If nonempty, open subsets of* Ω *have positive measure and suppose there exists a* $\bar{x} \in \Omega$ *such that* $f(\bar{x}, \cdot)$ *is continuous. If for all* $x \in \Omega$ *and* μ*-a.a.* $y \in \Omega$ *a kernel function* $f(x, y, \cdot) : Z \to \mathbb{R}^d$ *is* Y_+*-monotone and* $f(\bar{x}, y, \cdot) : Z \to \mathbb{R}^d$ *is strictly* Y_+*-monotone for* μ*-a.a.* $y \in \Omega$*, then an Urysohn operator* $\mathcal{F} : U \to C(\Omega)^d$ *is strictly* $C(\Omega)^d_+$*-monotone.*
3. *If nonempty, open subsets of* Ω *have positive measure and let* Y_+ *be solid. If for all* $x \in \Omega$ *and* μ*-a.a.* $y \in \Omega$ *a kernel function* $f(x, y, \cdot) : Z \to \mathbb{R}^d$ *is strongly* Y_+*-monotone, then an Urysohn operator* $\mathcal{F} : U \to C(\Omega)^d$ *is strongly* $C(\Omega)^d_+$*-monotone.*

Urysohn integral operators

Under the additional Hypothesis (U^1) the derivative of $\mathcal{F}$ exists as

$$D\mathcal{F}(u)v = \int_\Omega D_3 f(\cdot, y, u(y))v(y)\,\mathrm{d}\mu(y) \quad \text{for all } v \in C(\Omega)^d$$

in the Fréchet-sense (in $u \in U^\circ$); moreover $\mathcal{F}$ is of class C^1.

Corollary (monotonicity of $\mathcal{F}$)

Let (U^0), (U^1) hold on a Y_+-convex and open $Z \subseteq \mathbb{R}^d$. If $D_3 f(x, y, z)$ is Y_+-positive for all $x \in \Omega$, $z \in Z$ and μ-a.a. $y \in \Omega$, then $\mathcal{F}$ is $C(\Omega)^d_+$-monotone. In addition, if nonempty, open subsets of Ω have positive measure and moreover

1. *there exists a $\bar{x} \in \Omega$ so that $f(\bar{x}, \cdot)$ is continuous and for μ-a.a. $y \in \Omega$ and all $z, \bar{z} \in Z$, $z < \bar{z}$ the derivative $D_3 f(\bar{x}, y, z^*)$ is Y_+-injective for all $z^* \in \overline{z, \bar{z}}$, then $\mathcal{F}$ is strictly $C(\Omega)^d_+$-monotone,*
2. *Y_+ is solid and $D_3 f(x, y, z)$ is strongly Y_+-positive for all $x \in \Omega$, $z \in Z$ and μ-a.a. $y \in \Omega$, then $\mathcal{F}$ is strongly $C(\Omega)^d_+$-monotone.*

Nyström method

Applied to Urysohn operators (F) (with the Lebesgue measure $\mu = \lambda_\kappa$) one arrives at the *discrete Urysohn operator*

$$\mathcal{F}^n(u) := \sum_{j=0}^{q_n} w_j f(\cdot, \eta_j, u(\eta_j)). \qquad (F^n)$$

Let $Z \subseteq \mathbb{R}^d$ have nonempty interior. Assume that a kernel function $f : \Omega^2 \times Z \to \mathbb{R}^d$ fulfills the following continuity conditions:

Hypothesis

(NU^0) *$f : \Omega^2 \times Z \to \mathbb{R}^d$ exists as continuous function.*

Notice that (NU^0) implies the above Hypothesis (U^0). Furthermore, the discrete operator (**??**) allows the natural domains

$$U := \left\{u \in C(\Omega)^d : \; u(x) \in Z \text{ for all } x \in \Omega\right\}$$

and $U_n := \{u : \Omega_n \to Z\}$. In both cases, ($NU^0$) ensures that $\mathcal{F}^n$ is well-defined on U and U_n.

Theorem (properties of $\mathcal{F}^n$ on U)

Let Hypothesis (NU^0) hold.

1. *If for all $x \in \Omega$ and $\eta \in \Omega_n$ a kernel function $f(x, \eta, \cdot) : Z \to \mathbb{R}^d$ is Y_+-monotone and (Q_n), $n \in \mathbb{N}$, has nonnegative weights, then a discrete Urysohn operator $\mathcal{F}^n : U \to C(\Omega)^d$ is $C(\Omega)^d_+$-monotone.*
2. *If $\Omega = \overline{\Omega^\circ}$ and (Q_n) satisfying the net condition with eventually positive weights. For each $u, \bar{u} \in U$, $u \prec \bar{u}$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$: If for all $x \in \Omega$, $\eta \in \Omega_n$ a kernel function $f(x, \eta, \cdot) : Z \to \mathbb{R}^d$ is Y_+-monotone and $f(\bar{x}, \eta, \cdot)$ is strictly Y_+-monotone for one $\bar{x} \in \Omega$ and all $\eta \in \Omega_n$, then $\mathcal{F}^n(u) \prec \mathcal{F}^n(\bar{u})$.*
3. *If $\Omega = \overline{\Omega^\circ}$, solid Y_+ and (Q_n) satisfying the net condition with eventually positive weights. For each $u, \bar{u} \in U$, $u \prec \bar{u}$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$: If for all $x \in \Omega$, $\eta \in \Omega_n$ a kernel function $f(x, \eta, \cdot) : Z \to \mathbb{R}^d$ is strongly Y_+-monotone, then $\mathcal{F}^n(u) \lll \mathcal{F}^n(\bar{u})$.*

Nyström method

Theorem (necessary condition for monotonicity of $\mathcal{F}^n$ on U_n)

Let Hypothesis (NU^0) *hold. If a discrete Urysohn operator* $\mathcal{F}^n : U_n \to C(\Omega_n)^d$ *is strictly* $C(\Omega_n)^d_+$*-monotone on* U_n *for some* $n \in \mathbb{N}$ *and* $f(x, \eta, \cdot) : Z \to \mathbb{R}^d$ *is* Y_+*-monotone for all* $x \in \Omega$, $\eta \in \Omega_n$, *then quadrature rules* (Q_n) *has positive weights.*

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Thank you for attention