Monotonicity and discretization of integral operators

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Let \((X, \| \cdot \|)\) is a real Banach space. A nonempty closed and convex subset \(X_+ \subset X\) is called order cone, if \(\mathbb{R}_+ X_+ \subset X_+\) and \(X_+ \cap (-X_+) = \{0\}\) hold. Let us assume \(X_+ \neq \{0\}\) throughout and for elements \(x, \bar{x} \in X\) we write

\[
x \leq \bar{x} :\iff \bar{x} - x \in X_+ ,
\]

\[
x < \bar{x} :\iff \bar{x} - x \in X_+ \setminus \{0\} ,
\]

\[
x \ll \bar{x} :\iff \bar{x} - x \in X_+^o ,
\]

where the latter relation requires \(X_+^o \neq \emptyset\) (\(X_+^o\) is the interior of \(X_+)\) and one speaks of a solid cone \(X_+\).
Basic definitions

Let $X$ be a real Banach space, $X_+ \subset X$ a cone, $U \subseteq X$. A mapping $F : U \rightarrow X$ is called

- monotone, if
  \[ x < \bar{x} \Rightarrow F(x) \leq F(\bar{x}), \]

- strictly monotone, if
  \[ x < \bar{x} \Rightarrow F(x) < F(\bar{x}), \]

- strongly monotone, if
  \[ x < \bar{x} \Rightarrow F(x) \ll F(\bar{x}), \]

for all $x, \bar{x} \in U$.

In particular, a linear mapping $T : X \rightarrow X$ is

- monotone (then called positive), if $T(X_+ \setminus \{0\}) \subseteq X_+$,

- strictly monotone (then called strictly positive), if $T(X_+ \setminus \{0\}) \subseteq X_+ \setminus \{0\}$,

- strongly monotone (then called strongly positive), if $T(X_+ \setminus \{0\}) \subseteq X^+_\circ$. 
We equip a compact metric space $\Omega$ with a $\sigma$-algebra $\mathcal{A}$ (containing the Borel sets) and a measure $\mu$ such that $(\Omega, \mathcal{A}, \mu)$ is a measure space satisfying $\mu(\Omega) < \infty$. The set $C(\Omega)^d$ of all continuous functions $u : \Omega \to \mathbb{R}^d$ is a real Banach space with norm $\|u\|_{\infty} := \max_{x \in \Omega} |u(x)|$. Moreover,

$$C(\Omega)^d_+ := \{u \in C(\Omega)^d : u(x) \in Y_+ \text{ for all } x \in \Omega\}$$

abbreviates the set of continuous functions having values in the cone $Y_+ \subset \mathbb{R}^d$.

**Lemma**

*The set $C(\Omega)^d_+$ is a cone, which is solid, provided $Y_+$ is.*
Basic definitions

Having identified $C(\Omega)_+^d$ as (solid) cone, we introduce the relations

\[
\begin{align*}
u \preceq \bar{u} & :\iff \bar{u} - u \in C(\Omega)_+^d, \\
u \prec \bar{u} & :\iff \bar{u} - u \in C(\Omega)_+^d \setminus \{0\}, \\
u \ll \bar{u} & :\iff \bar{u} - u \in (C(\Omega)_+^d)^\circ \quad \text{for all } u, \bar{u} \in C(\Omega)^d.
\end{align*}
\]

Lemma

The following holds for $u, \bar{u} \in C(\Omega)^d$:

1. $u \preceq \bar{u} \iff u(x) \leq \bar{u}(x)$ for all $x \in \Omega \iff \langle u(x), y' \rangle \leq \langle \bar{u}(x), y' \rangle$ for all $x \in \Omega$ and $y' \in Y'_+$.

2. $u \prec \bar{u} \iff u(x) \leq \bar{u}(x)$ for all $x \in \Omega$ and $u(x_0) < \bar{u}(x_0)$ for some $x_0 \in \Omega$.

3. If $Y_+$ is solid, then one has

$u \ll \bar{u} \iff u(x) \ll \bar{u}(x)$ for all $x \in \Omega \iff \langle u(x), y' \rangle < \langle \bar{u}(x), y' \rangle$ for all $x \in \Omega$, $y' \in Y'_+ \setminus \{0\}$. 


Now we want to provide sufficient conditions for Fredholm integral operators

\[ \mathcal{K}u := \int_{\Omega} K(\cdot, y)u(y) \, d\mu(y) \]

to preserve order relations. In essence, we demonstrate that monotonicity (positivity) properties of the kernel functions carry over to the integral operators.

We consider Fredholm operators under assumption for \( K : \Omega \times \Omega \rightarrow L(\mathbb{R}^d) \):

**Hypothesis**

\[ (L) \quad K(x, \cdot) : \Omega \rightarrow L(\mathbb{R}^d) \text{ is } \mu\text{-measurable for all } x \in \Omega \text{ with} \]

\[ \sup_{x \in \Omega} \int_{\Omega} |K(x, y)| \, d\mu(y) < \infty \text{ and} \]

\[ \lim_{x \to x_0} \int_{\Omega} |K(x, y) - K(x_0, y)| \, d\mu(y) = 0 \text{ for all } x_0 \in \Omega, \]

which yield that \( \mathcal{K} \in L(C(\Omega)^d) \).
Fredholm integral operators

Theorem (positivity of $\mathcal{K}$ on $C(\Omega)^d$)

Let Hypothesis (L) hold. If $K(x, y)$ is $Y_+$-positive for all $x \in \Omega$ and $\mu$-a.a. $y \in \Omega$, then a Fredholm operator $\mathcal{K} \in L(C(\Omega)^d)$ is $C(\Omega)^d_+$-positive.

Furthermore we denote $T \in L(X)$ as $X_+$-injective, provided its kernel satisfies $\mathcal{N}(T) \cap X_+ = \{0\}$.

Theorem (strictly positivity of $\mathcal{K}$ on $C(\Omega)^d$)

Let Hypothesis (L) hold and $K(x, y)$ is $Y_+$-positive for all $x \in \Omega$ and $\mu$-a.a. $y \in \Omega$. If nonempty, open subsets of $\Omega$ have positive measure, there exists a $\bar{x} \in \Omega$ so that $K(\bar{x}, \cdot)$ is continuous and $K(\bar{x}, y)$ is $Y_+$-injective for $\mu$-a.a. $y \in \Omega$, then $\mathcal{K}$ strictly $C(\Omega)^d_+$-positive.

Theorem (strongly positivity of $\mathcal{K}$ on $C(\Omega)^d$)

Let Hypothesis (L) holds. If nonempty, open subsets of $\Omega$ have positive measure, $Y_+$ is solid and $K(x, y)$ is strongly $Y_+$-positive for all $x \in \Omega$ and $\mu$-almost all $y \in \Omega$, then $\mathcal{K}$ is strongly $C(\Omega)^d_+$-positive.
Nyström methods

We suppose $\Omega \subset \mathbb{R}^\kappa$ is compact with positive Lebesgue measure $\lambda_\kappa(\Omega) > 0$. Given a continuous function $u : \Omega \to \mathbb{R}^d$, consider the representation

$$\int_\Omega u(y) \, dy = \sum_{j=0}^{q_n} w_j u(\eta_j) + E_n(u)$$

with a sequence $(q_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$, nodes from a finite set $\Omega_n := \{\eta_0, \ldots, \eta_{q_n}\} \subseteq \Omega$, weights $w_j \in \mathbb{R}$ such that the remainder (error term) satisfies $\lim_{n \to \infty} E_n(u) = 0$. Such schemes are called convergent.

We say that an integration rule $(Q_n)$ fulfills the net condition, if

$$\forall \varepsilon > 0 : \exists n_0 \in \mathbb{N} : \Omega \subseteq \bigcup_{j=0}^{q_n} B_\varepsilon(\eta_j) \text{ for all } n \geq n_0(\varepsilon).$$
Nyström methods

A natural way to evaluate the Lebesgue integral in the Fredholm operator is to apply a rule \((Q_n)\).

Hypothesis

Assume that a kernel \(K : \Omega \times \Omega \rightarrow L(\mathbb{R}^d)\) fulfills:

\((NL)\) \quad K(\cdot, y) : \Omega \rightarrow L(\mathbb{R}^d)\) is continuous for all \(y \in \Omega\).

This leads to the (spatially) discrete Fredholm integral operator

\[
\mathcal{K}^n u := \sum_{j=0}^{q_n} w_j K(\cdot, \eta_j) u(\eta_j).
\]

There are two natural choices for the domain of \(\mathcal{K}^n\), namely a spatially continuous one \(C(\Omega)^d\) and the spatially discrete function space

\[
C(\Omega_n)^d = \{ u : \Omega_n \rightarrow \mathbb{R}^d \};
\]

both are equipped with the max-norm. In each case, \((NL)\) suffices to obtain that \(\mathcal{K}^n\) is well-defined.
Nyström methods

\[ \mathcal{K}^n u := \sum_{j=0}^{q_n} w_j K(\cdot, \eta_j) u(\eta_j) \]

Remark (\(\mathcal{K}^n\) on the domain \(C(\Omega_n)^d\))

Suppose that \((Q_n)\) has nonnegative weights. Then previous Theorem (positivity of \(\mathcal{K}\) on \(C(\Omega)^d\)) applies in the special case \(\mu = \mu_n\) and guarantees that positivity of \(\mathcal{K}^n \in L(C(\Omega_n)^d)\) or \(\mathcal{K}^n \in L(C(\Omega_n)^d, C(\Omega)^d)\) holds literally with the assumption "\(\mu\)-a.a. \(y \in \Omega\)" replaced by "all \(y \in \Omega_n\)".

Remark (\(\mathcal{K}^n\) on the domain \(C(\Omega)^d\))

On the domain \(C(\Omega)^d\) one cannot expect \(\mathcal{K}^n\) to be strictly or strongly positive. This is due to the fact that \(C(\Omega)^d \setminus \{0\}\) contains functions \(u\) vanishing except from being positive on arbitrarily small domains disjoint from \(\Omega_n\). Hence, they are not captured by the Nyström grid \(\Omega_n\), that is, \(u|_{\Omega_n} = 0\) although \(u \neq 0\). Consequently, one has \(\mathcal{K}^n u = 0\).
### Nyström methods

**Theorem (positivity of $\mathcal{K}^n$ on $C(\Omega)^d$)**

Let $K(\cdot, y) : \Omega \to L(\mathbb{R}^d)$ is continuous for all $y \in \Omega$ hold.

1. If $(Q_n)$, $n \in \mathbb{N}$, have nonnegative weights and $K(x, \eta)$ is $Y_+$-positive for all $x \in \Omega$, $\eta \in \Omega_n$, then $\mathcal{K}^n \in L(C(\Omega)^d)$ is $C(\Omega)^d_+$-positive.

2. If $(Q_n)$, $n \in \mathbb{N}$, have positive weights, $Y_+$ is solid and $K(x, \eta) Y_+^\circ \subseteq Y_+^\circ$ for all $x \in \Omega$, $\eta \in \Omega_n$, then $\mathcal{K}^n(C(\Omega)^d_+)^\circ \subseteq (C(\Omega)^d_+)^\circ$.
Theorem (eventually positivity of $K^n$ on $C(\Omega)^d$)

Let $K(\cdot, y) : \Omega \to L(\mathbb{R}^d)$ is continuous for all $y \in \Omega$, $\Omega = \overline{\Omega^o}$ and $(Q_n)$ have eventually positive weights and the net condition hold, then for each $u \in C(\Omega)^d$ with $0 \prec u$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$:

1. If $K(x, \eta)$ is $Y_+$-positive for all $x \in \Omega$, $\eta \in \Omega_n$ and $K(\bar{x}, \eta)$ is $Y_+$-injective for one $\bar{x} \in \Omega$ and all $\eta \in \Omega_n$, then $0 \prec K^n u$.

2. If $Y_+$ is solid and $K(x, \eta)$ is strongly $Y_+$-positive for all $x \in \Omega$, $\eta \in \Omega_n$, then $0 \ll K^n u$. 

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In projection method we consider a linear projection $\Pi_n \in L(C(\Omega)^d, X_n^d)$, where $X_n^d$ is a subspace of $C(\Omega)^d$ with $X_n = \text{lin}\{\phi_1, \ldots, \phi_{d_n}\}$. The spatial discretizations of an integral operator $K \in L(X(\Omega)^d)$ become

$$K^n := \Pi_n K.$$ 

One of projection method is a collocation method, in which pairwise different collocation points $x_1, \ldots, x_{d_n} \in \Omega$ satisfy the interpolation conditions and follow to collocation matrix

$$[\phi_i(x_j)]_{i,j=1}^{d_n}$$

which is denoted by $P_n$. 
Theorem (positivity of $\Pi_n$ on $C(\Omega)^d$)

If all the functions

$$\sigma_i : \Omega \to \mathbb{R}, \quad \sigma_i(x) := \sum_{j=1}^{d_n} (P_n^{-1})_{ij} \phi_j(x) \quad \text{for all } 1 \leq i \leq d_n$$

have nonnegative values, then the following hold:

1. $\Pi_n$ is $C(\Omega)^d_+$-positive.
2. If additionally $Y_+$ is solid and

$$\forall x \in \Omega : \exists i_0 \in \{1, \ldots, d_n\} : \sigma_{i_0}(x) > 0$$

holds, then $\Pi_n(C(\Omega)^d_+)^\circ \subset (C(\Omega)^d_+)^\circ$. 
\[ P_n = [\phi_i(x_j)]_{i,j=1}^{d_n} \quad \sigma_i(x) := \sum_{j=1}^{d_n} (P_n^{-1})_{ij} \phi_j(x) \quad \text{for all } 1 \leq i \leq d_n \]

On the other hand, the applicability of Theorem (positivity of \( \Pi_n \) on \( C(\Omega)^d \)) is hindered by the following fact: Many bases \( \phi_1, \ldots, \phi_{d_n} \) (e.g. \( B \)-splines, Bernstein polynomials, etc.) consist of functions having nonnegative values yielding a nonnegative collocation matrix \( P_n \). Thus, \( P_n^{-1} \) has nonnegative entries, if and only if \( P_n \) is a monomial matrix, i.e. every column/row contains exactly one positive element.

Positive projections have:
- piecewise linear collocation;
- polynomial interpolation;
- cubic spline.

Projection which is not positive: quadratic splines.
Corollary (positivity of $K^n$ on $C(\Omega)^d$)

Let Hypothesis (L) hold and all the functions

$$\sigma_i : \Omega \rightarrow \mathbb{R}, \quad \sigma_i(x) := \sum_{j=1}^{d_n} (P_{n}^{-1})_{ij} \phi_j(x) \quad \text{for all } 1 \leq i \leq d_n$$

have nonnegative values.

1. If $K$ is $C(\Omega)^d_+$-positive, then $K^n = \prod_n K \in L(C(\Omega)^d, X_n)$ and $K\prod_n \in L(C(\Omega)^d)$ are $C(\Omega)^d_+$-positive.

2. If $K$ is strongly $C(\Omega)^d_+$-positive, $Y_+$ is solid and

$$\forall x \in \Omega : \exists i_0 \in \{1, \ldots, d_n\} : \sigma_{i_0}(x) > 0$$

holds, then $K^n = \prod_n K \in L(C(\Omega)^d, X_n)$ and $K\prod_n \in L(C(\Omega)^d)$ are strongly $C(\Omega)^d_+$-positive.
Urysohn integral operators

\[ \mathcal{F} : U \to C(\Omega)^d, \quad \mathcal{F}(u) := \int_{\Omega} f(\cdot, y, u(y)) \, d\mu(y), \quad (F) \]

\[ U := \{ u \in C(\Omega)^d : u(x) \in Z \text{ for all } x \in \Omega \}. \]

Let \( Z \subseteq \mathbb{R}^d \) be nonempty. Assume for a kernel function \( f : \Omega^2 \times Z \to \mathbb{R}^d \):

**Hypothesis**

\((U^0)\) \( f(x, \cdot, z) : \Omega \to \mathbb{R}^d \) is \( \mu \)-measurable for all \( x \in \Omega \), \( z \in Z \), for every \( r > 0 \) there exists a \( \mu \)-measurable function \( \beta_r^0 : \Omega^2 \to \mathbb{R}_+ \) satisfying

\[ \sup_{x \in \Omega} \int_{\Omega} \beta_r^0(x, y) \, d\mu(y) < \infty, \]

such that for \( \mu \)-a.a. \( y \in \Omega \) it is \( |f(x, y, z)| \leq \beta_r^0(x, y) \) for all \( x \in \Omega \), \( z \in Z \cap \overline{B}_r(0) \) and \( f(\cdot, y, \cdot) : \Omega \times Z \to L(\mathbb{R}^d) \) exists as continuous function for \( \mu \)-a.a. \( y \in \Omega \). Furthermore, for every \( r > 0 \) there exist a \( \mu \)-measurable function \( \gamma_r^0 : \Omega^3 \to \mathbb{R}_+ \) satisfying

\[ \lim_{x \to x_0} \int_{\Omega} \gamma_r^0(x, x_0, y) \, d\mu(y) = 0 \quad \text{for all } x_0 \in \Omega, \]

such that for \( \mu \)-a.a. \( y \in \Omega \) one has \( |f(x, y, z) - f(\bar{x}, y, z)| \leq \gamma_r^0(x, \bar{x}, y) \) for all \( x, \bar{x} \in \Omega \), \( z \in Z \cap \overline{B}_r(0) \).
Theorem (properties of $\mathcal{F}$)

Let Hypothesis $(U^0)$ hold.

1. If for all $x \in \Omega$ and $\mu$-a.a. $y \in \Omega$ a kernel function $f(x, y, \cdot) : Z \to \mathbb{R}^d$ is $Y_+$-monotone, then an Urysohn operator $\mathcal{F} : U \to C(\Omega)^d$ is $C(\Omega)^+_d$-monotone.

2. If nonempty, open subsets of $\Omega$ have positive measure and suppose there exists a $\bar{x} \in \Omega$ such that $f(\bar{x}, \cdot)$ is continuous. If for all $x \in \Omega$ and $\mu$-a.a. $y \in \Omega$ a kernel function $f(x, y, \cdot) : Z \to \mathbb{R}^d$ is $Y_+$-monotone and $f(\bar{x}, y, \cdot) : Z \to \mathbb{R}^d$ is strictly $Y_+$-monotone for $\mu$-a.a. $y \in \Omega$, then an Urysohn operator $\mathcal{F} : U \to C(\Omega)^d$ is strictly $C(\Omega)^+_d$-monotone.

3. If nonempty, open subsets of $\Omega$ have positive measure and let $Y_+$ be solid. If for all $x \in \Omega$ and $\mu$-a.a. $y \in \Omega$ a kernel function $f(x, y, \cdot) : Z \to \mathbb{R}^d$ is strongly $Y_+$-monotone, then an Urysohn operator $\mathcal{F} : U \to C(\Omega)^d$ is strongly $C(\Omega)^+_d$-monotone.
Under the additional Hypothesis \((U^1)\) the derivative of \(F\) exists as

\[
D F(u)v = \int_{\Omega} D_3 f(\cdot, y, u(y)) v(y) \, d\mu(y) \quad \text{for all } v \in C(\Omega)^d
\]

in the Fréchet-sense \((u \in U^o)\); moreover \(F\) is of class \(C^1\).

**Corollary (monotonicity of \(F\))**

Let \((U^0), (U^1)\) hold on a \(Y_+\)-convex and open \(Z \subseteq \mathbb{R}^d\). If \(D_3 f(x, y, z)\) is \(Y_+\)-positive for all \(x \in \Omega, z \in Z\) and \(\mu\)-a.a. \(y \in \Omega\), then \(F\) is \(C(\Omega)^d_+\)-monotone. In addition, if nonempty, open subsets of \(\Omega\) have positive measure and moreover

1. there exists a \(\bar{x} \in \Omega\) so that \(f(\bar{x}, \cdot)\) is continuous and for \(\mu\)-a.a. \(y \in \Omega\) and all \(z, \bar{z} \in Z, z < \bar{z}\) the derivative \(D_3 f(\bar{x}, y, z^*)\) is \(Y_+\)-injective for all \(z^* \in \overline{z, \bar{z}}\), then \(F\) is strictly \(C(\Omega)^d_+\)-monotone,

2. \(Y_+\) is solid and \(D_3 f(x, y, z)\) is strongly \(Y_+\)-positive for all \(x \in \Omega, z \in Z\) and \(\mu\)-a.a. \(y \in \Omega\), then \(F\) is strongly \(C(\Omega)^d_+\)-monotone.
Nyström method

Applied to Urysohn operators \((F)\) (with the Lebesgue measure \(\mu = \lambda_\kappa\)) one arrives at the \textit{discrete Urysohn operator}

\[
\mathcal{F}^n(u) := \sum_{j=0}^{q_n} w_j f(\cdot, \eta_j, u(\eta_j)).
\]

\((F^n)\)

Let \(Z \subseteq \mathbb{R}^d\) have nonempty interior. Assume that a kernel function \(f : \Omega^2 \times Z \rightarrow \mathbb{R}^d\) fulfills the following continuity conditions:

**Hypothesis**

\((NU^0)\) \(f : \Omega^2 \times Z \rightarrow \mathbb{R}^d\) exists as continuous function.

Notice that \((NU^0)\) implies the above Hypothesis \((U^0)\). Furthermore, the discrete operator \((??)\) allows the natural domains

\[
U := \{ u \in C(\Omega)^d : u(x) \in Z \text{ for all } x \in \Omega \}
\]

and \(U_n := \{ u : \Omega_n \rightarrow Z \}\). In both cases, \((NU^0)\) ensures that \(\mathcal{F}^n\) is well-defined on \(U\) and \(U_n\).
Theorem (properties of $\mathcal{F}^n$ on $U$)

Let Hypothesis $(NU^0)$ hold.

1. If for all $x \in \Omega$ and $\eta \in \Omega_n$ a kernel function $f(x, \eta, \cdot): Z \to \mathbb{R}^d$ is $Y_+^\ast$-monotone and $(Q_n)$, $n \in \mathbb{N}$, has nonnegative weights, then a discrete Urysohn operator $\mathcal{F}^n: U \to C(\Omega)^d$ is $C(\Omega)^d_+^\ast$-monotone.

2. If $\Omega = \overline{\Omega}$ and $(Q_n)$ satisfying the net condition with eventually positive weights. For each $u, \bar{u} \in U$, $u \prec \bar{u}$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$: If for all $x \in \Omega$, $\eta \in \Omega_n$ a kernel function $f(x, \eta, \cdot): Z \to \mathbb{R}^d$ is $Y_+^\ast$-monotone and $f(\bar{x}, \eta, \cdot)$ is strictly $Y_+^\ast$-monotone for one $\bar{x} \in \Omega$ and all $\eta \in \Omega_n$, then $\mathcal{F}^n(u) \prec \mathcal{F}^n(\bar{u})$.

3. If $\Omega = \overline{\Omega}$, solid $Y_+$ and $(Q_n)$ satisfying the net condition with eventually positive weights. For each $u, \bar{u} \in U$, $u \prec \bar{u}$ there exists a $N \in \mathbb{N}$ such that one has for $n \geq N$: If for all $x \in \Omega$, $\eta \in \Omega_n$ a kernel function $f(x, \eta, \cdot): Z \to \mathbb{R}^d$ is strongly $Y_+^\ast$-monotone, then $\mathcal{F}^n(u) \preccurlyeq \mathcal{F}^n(\bar{u})$. 
Theorem (necessary condition for monotonicity of $\mathcal{F}^n$ on $U_n$)

Let Hypothesis ($NU^0$) hold. If a discrete Urysohn operator $\mathcal{F}^n : U_n \rightarrow C(\Omega_n)^d$ is strictly $C(\Omega_n)^d_+-$monotone on $U_n$ for some $n \in \mathbb{N}$ and $f(x, \eta, \cdot) : Z \rightarrow \mathbb{R}^d$ is $Y_+-$monotone for all $x \in \Omega$, $\eta \in \Omega_n$, then quadrature rules ($Q_n$) has positive weights.


Thank you for attention