Nonlocal heat equations with generalized fractional Laplacian

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Analysis in Tatra Seminar for researchers, Małe Ciche, September 7-11, 2022

Motivation

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- Diffusion-reaction model

$$u_t + (-\Delta)^{\alpha/2} u + c(t,x)u = 0.$$

M. Yamamoto, Asymptotic expansion of solutions to the dissipative equation with fractional Laplacian, SIAM J. Math. Anal. 44(6) (2012) 3786—3805.

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- Drift-diffusion model for semiconductor

$$u_t + (-\Delta u)^{\theta/2} - \nabla \cdot (u \nabla \psi) = 0, \qquad -\Delta \psi = u.$$

M. Yamamoto, Y. Sugiyama, Asymptotic expansion of solutions to the drift–diffusion equation with fractional dissipation, Nonlinear Analysis: Theory, Methods and Applications, Volume 14 (2016) 57–87.

Various definitions

- M. Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fract. Calc. Appl. Anal. **20** 2017, 7-51.

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- R. Servadei, E. Valdinoci, *On the spectrum of different two fractional operators*, Proc. Roy. Soc. Edinburgh Sect. A **144** (2014), 831–855.

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If Ω being of class C^2 or Ω being cuboid, we have $-\Delta = (-\Delta)_s$, where $(-\Delta)_s\colon H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$ means the strong Dirichlet-Laplace operator.

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$$\operatorname{Dom} g(-\Delta) := \left\{ u \in L^2(\Omega) : \sum_{n=1}^{\infty} g(\lambda_n)^2 | \langle u, e_n \rangle |^2 < \infty \right\}.$$

Poisson equation with generalized laplacian

We obtained some results for Poisson equation with generalized fractional laplacian (I.K., B.Przeradzki, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **115** (2), Paper No. 58 (2021))

$$g(-\Delta)u = f(x,u).$$

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$$u_n'(t) + g(\lambda_n)u_n(t) = f_n(t)(u),$$

where

$$f_n(t)(u) := \int_\Omega f(t,y,u(t,y))e_n(x)\,\mathrm{d}y.$$

Heat equation with generalized Laplacian

Hence

$$u_n(t) = u_{n,0}\mathrm{e}^{-g(\lambda_n)t} + \int\limits_0^t \mathrm{e}^{-g(\lambda_n)(t-s)} f_n(s)(u)\,\mathrm{d}s$$

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Fix $T > 0$ and search for solutions on the interval $[0, T]$. Let X be a Banach space of sequences of real continuous functions $u_n : [0, T] \to \mathbb{R}$, $n \in \mathbb{N}$, such that

$$\|u\|^2 := \sum_{n=1}^{\infty} \sup_{t\in[0,T]} |u_n(t)|^2 < \infty,$$

where $u = (u_n)_{n\in\mathbb{N}}$.

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$$(S(v))_n(t) := v_{0,n} \mathrm{e}^{-g(\lambda_n)t} + \int_0^t \mathrm{e}^{-g(\lambda_n)(t-s)} f_n(s)(v)\, \mathrm{d}s$$

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Theorem 5

The initial-boundary value problem for heat equation with $g(-\Delta)$ has a solution.

Asymptotic behaviour of solutions

Theorem 6

$$u_t + g(-\Delta)u = f(x,u), \quad u(t,\cdot)|_{\partial\Omega} = 0, \quad u(0,\cdot) = u_0 \in L^2(\Omega), \tag{1}$$

$$g(-\Delta)u = f(x,u), \qquad u|_{\partial\Omega} = 0. \tag{2}$$

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Suppose f satisfies the Lipschitz condition

$$|f(x,u) - f(x,v)| \le L|u - v|$$

for any $u, v \in \mathbb{R}$ and a.e. $x \in \Omega$, where $L < \beta := \inf_{n\in\mathbb{N}} g(\lambda_n)$. Then (2) has the unique solution w and

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$$\lim_{t\to+\infty} \|u(t,\cdot) - w\| = 0.$$

Applications of Semigroups

Let g be a real function defined on the spectrum $(\lambda_n)_{n\in\mathbb{N}}$ of the Dirichlet Laplacian such that $\lim_{n\to\infty} g(\lambda_n) = +\infty$.

Definition

$$T(t)u = \sum_{n=1}^{\infty} \mathrm{e}^{-g(\lambda_n)t} \langle u, e_n \rangle e_n \qquad \text{for } u \in L^2(\Omega).$$

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Proposition

The family $\{T(t)\}_{t\geq 0}$ is C_0-semigroup in $L^2(\Omega)$ with infinitesimal generator $-g(-\Delta)$.

Applications of Semigroups

We consider the following semilinear initial value problem

$$\boldsymbol{u}'(t) = A\boldsymbol{u}(t) + \boldsymbol{f}(t, \boldsymbol{u}(t)), \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{3}$$

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If A is the generator of a compact semigroup $\{T(t)\}_{t\geq 0}$, $\boldsymbol{f}\colon [0,\infty) \times U \to X$ is continuous, where $U \subset X$ is open then for every $\boldsymbol{u}_0 \in U$ there exists a $t_1 \in (0,\infty)$ (3) has a mild solution $\boldsymbol{u} \in C([0, t_1,], X)$.

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Theorem

Let A be the infinitesimal generator of a compact semigroup $\{T(t)\}_{t\geq 0}$. Let $\boldsymbol{f}\colon [0, +\infty) \times X \to X$ be continuous and maps bounded sets in $[0, +\infty) \times X$ into bounded sets in X. Then for every $u_0 \in X$ the initial value problem (3) has a global solution $\boldsymbol{u} \in C([0, +\infty), X)$ if there exist two locally integrable functions $k_1, k_2\colon [0, +\infty) \to [0, +\infty)$ such that $\|\boldsymbol{f}(t, \boldsymbol{u})\| \leq k_1(t)\, \|\boldsymbol{u}\| + k_2(t)$ for $t \in [0, +\infty), \boldsymbol{u} \in X$.

Applications of Semigroups

For the operator $A = -g(-\Delta)$, and mapping $f(t,u) = f(t,\cdot,u(\cdot))$ for $t \geq 0$, $u \in L^2(\Omega)$, the initial-boundary value problem

$$u_t + g(-\Delta)u = f(t,x,u), \quad u(t,\cdot)|_{\partial\Omega} = 0, \quad u(0,\cdot) = u_0 \in L^2(\Omega) \tag{4}$$

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Theorem 7

Under the above assumptions, the problem (4) has a global mild solution.

Numerical simulations

We will use the simplest method for finding an approximate solution – a partial sum of the Fourier series

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t) e_n(x).$$

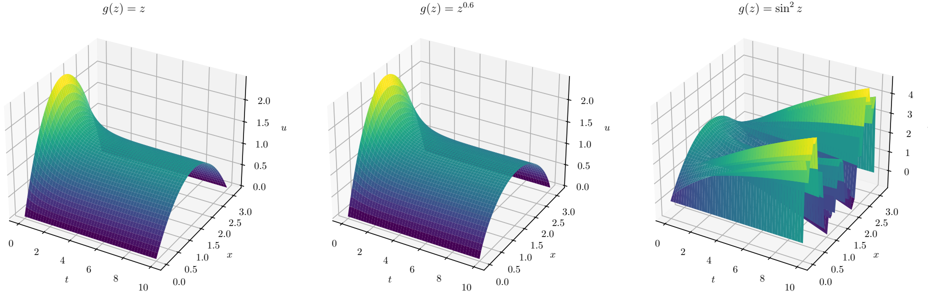
One can find the explicit formulas for all functions if the right-hand side f has the simple form $bu + f(x)$ and $\Omega := (0,\pi) \subset \mathbb{R}$. We have

$$e_n(x) := \sqrt{\frac{2}{\pi}} \sin nx,$$

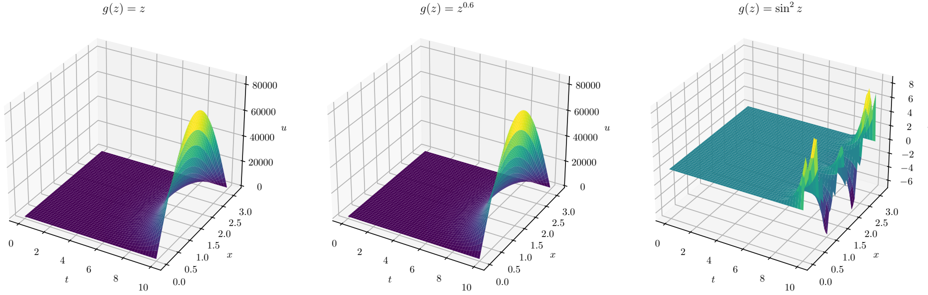
$$u_n(t) := \left(u_{n,0} - \frac{f_n}{g(n^2) - b} \right) \exp(-(g(n^2) - b)t) + \frac{f_n}{g(n^2) - b},$$

where f_n is the n-th Fourier coefficient of $x \mapsto f(x)$.

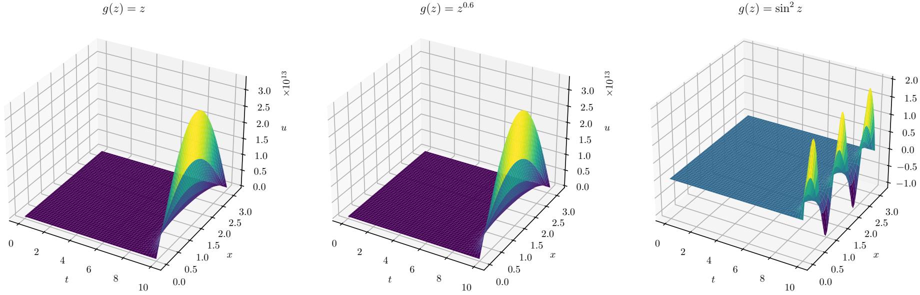
Solutions to our problem for $f(x) \equiv 1$, $b = 0$, $u_0(x) = x(\pi - x)$



Solutions to our problem for $f(x) \equiv 1$, $b = 2$, $u_0(x) = x(\pi - x)$



Solutions to our problem for $f(x) = 1 - \cos 2x$, $b = 4$, $u_0(x) = x(\pi - x)$



Thank you for your attention.