Sharp estimates for the first eigenvalue of the Robin p-Laplacian as p goes to 1

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First eigenvalue of the p-Laplacian

$$\lambda(\Omega,p,\beta) = \min_{v\in W^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p + \beta\int_{\partial\Omega}|v|^p}{\int_\Omega |v|^p}$$

$\lambda > 0$ if $\beta > 0$

$\lambda < 0$ if $\beta < 0$

Ω bounded, Lipschitz domain in $\mathbb{R}^N$

$p \in]1,+\infty[$

$\beta \in \mathbb{R}$

A minimizer u solves

$$\underbrace{\begin{cases} -\Delta_p u = \lambda(\Omega,p,\beta)|u|^{p-2}u \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 \end{cases}}_{\text{Robin b.c.}}$$

$\beta = +\infty$ Dirichlet ; $u|_{\partial\Omega} = 0$

$\beta = 0$ Neumann trivial eigenvalue

$\beta \to -\infty$ $\lambda(\Omega,p,\beta) \to -\infty$

Shape optimization problem

Given $p \in]1,+\infty[$, $\beta \in \mathbb{R}$, to study

$$\inf_{|\Omega|=k} \lambda(\Omega, p, \beta) \qquad \sup_{|\Omega|=k} \lambda(\Omega, p, \beta)$$

Striking difference between the cases

$\beta > 0$ and $\beta < 0$

$\beta \geq 0$

$$\sup_{|\Omega| = k} \lambda(\Omega, p, \beta) = +\infty$$

Faber-Krahn inequality

$$\lambda(\Omega, p, \beta) \geq \lambda(B, p, \beta)$$

B ball s.t. $|B| = |\Omega|$

$p = N = 2$ Bossel 1988;

$p = 2, N \geq 2$ Daners 2006;

$p \in]1, +\infty[$ Bucur-Daners 2010; Dai-Fu 2011

$\beta < 0$

$$\inf_{|\Omega| = k} \lambda(\Omega, p, \beta) = -\infty \qquad \lambda(\Omega, p, \beta) \le \beta \frac{P(\Omega)}{|\Omega|}$$

Bareket conjecture: the ball maximizes $\lambda(\Omega, 2, \beta)$ among the smooth domains of given volume

$|\beta|$ large: false   Freitas - Krejčiřík 2015, $p = 2$
Kovařík - Pankrashkin 2017 $1 < p < +\infty$, $N \geq 1$

$|\beta|$ small: true   Freitas - Krejčiřík 2015, $p = N = 2$
Kovařík - Pankrashkin 2017 $1 < p < +\infty$, $N \geq 1$, star-shaped domains

Open problem: to find the optimal β such that the conjecture is true

$\beta < 0$

Asymptotic formula:

$$\lambda(\Omega, \beta, p) = -(p-1)|\beta|^{\frac{p}{p-1}} - (n-1)H_{max}|\beta| + o(\beta), \quad \beta \to -\infty$$

H_{max} = maximum mean curvature of $\partial\Omega \in C^{1,1}$

$$\lambda(B_\rho(0)) - \lambda\left(B_R(0) \setminus \overline{B_r(0)}\right) = (n-1)|\beta| \underbrace{\left[\frac{1}{R} - \frac{1}{\rho}\right]}_{<0} + o(\beta)$$

$\beta \to -\infty$

$\rho^n = R^n - r^n$ $\qquad$ $r < \rho < R$

$\beta < 0$ If the perimeter, instead of the volume, is fixed, it holds that:

$$\lambda(\Omega, p, \beta) \leq \lambda(\tilde{B}, p, \beta) \qquad P_{er}(\Omega) = P_{er}(\tilde{B})$$

$N = p = 2$, smooth domains: Antunes, Freitas, Krejčiřík 2017

$p \in]1, +\infty[$, $N \geq 2$, convex domains: Bucur-Ferone-Nitsch-Trombetti: 2019

Objective: to study what happens as $p \to 1$

$$J(u) = \frac{|\nabla u|(\Omega) + \min\{\beta, 1\} \int_{\partial\Omega} |u|}{\int_{\Omega} |u|}, \qquad u \in BV(\Omega)$$

If $\beta \geq 1$, $J(u) = \dfrac{|\nabla u|(\mathbb{R}^N)}{\int_{\Omega} |u|}$

$$\Lambda(\Omega, \beta) = \inf_{u \in BV(\Omega)} J(u)$$

Remark 1 Let $\beta \geq 1$

$$\Lambda(\Omega,\beta) = \inf_{u \in BV(\Omega)} \frac{|\nabla u|(\mathbb{R}^N)}{\int_\Omega |u|}$$

$$= \inf_{E \subseteq \Omega} \frac{Per(E)}{|E|} = h(\Omega)$$

Cheeger constant

Cheeger inequality:

$$\lambda(\Omega, p, +\infty) \geq \left(\frac{h(\Omega)}{p}\right)^p$$

Remark 2 $\beta \geq 1$

$$\lambda(\Omega,1,\beta) = \inf_{BV(\Omega)} \frac{|\nabla u|(\Omega) + \beta \int_{\partial\Omega} |u|}{\int_\Omega |u|}$$

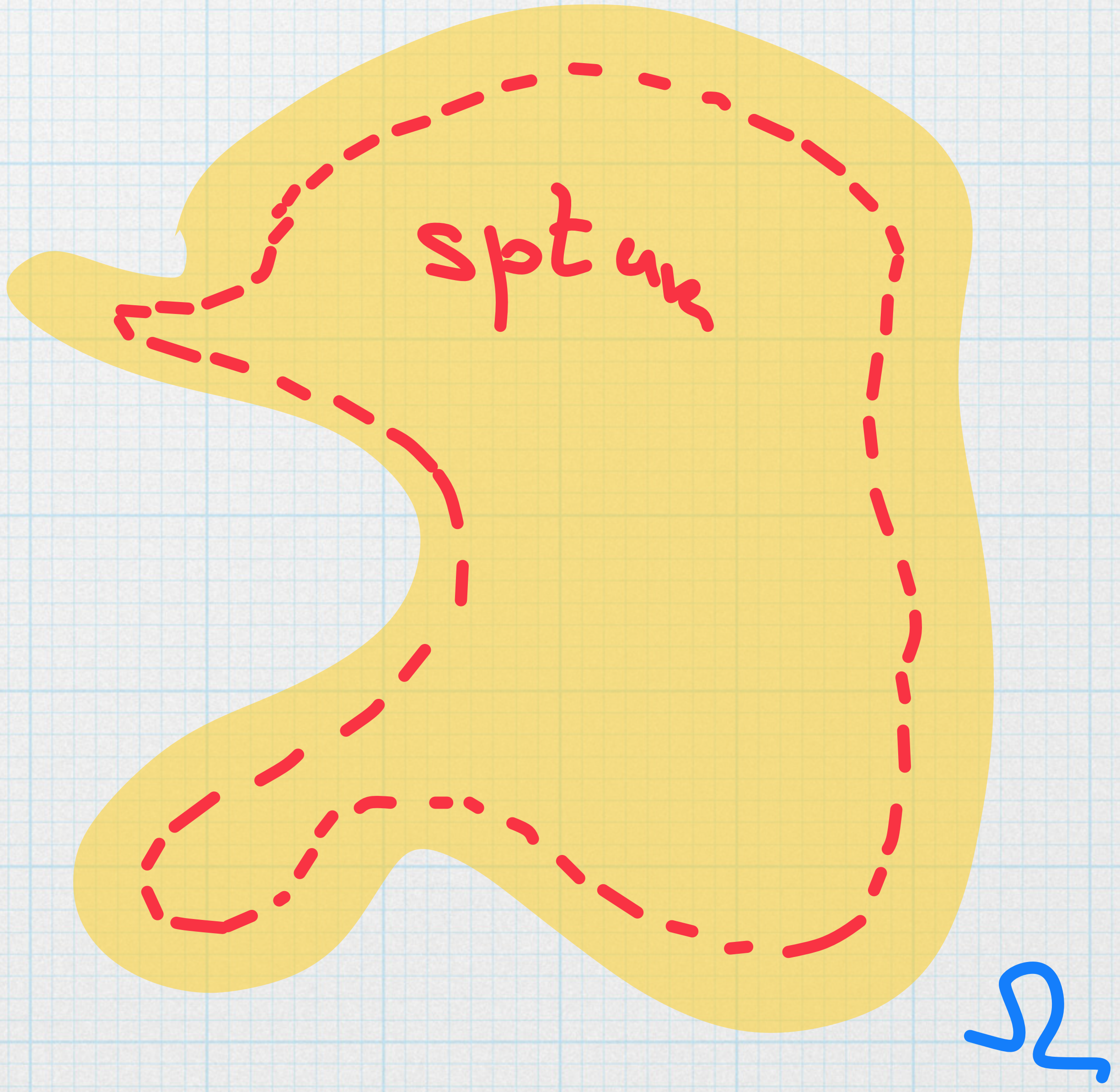
$$\Lambda(\Omega,\beta) = \inf_{BV(\Omega)} \frac{|\nabla u|(\Omega) + \int_{\partial\Omega} |u|}{\int_\Omega |u|}$$

If $\beta \geq 1$, then

$$\lambda(\Omega,1,\beta) = \Lambda(\Omega,\beta)$$

If $u \in BV(\Omega)$, $\exists\, u_k \in C_0^\infty(\Omega)$: $\begin{cases} u_k \longrightarrow u \text{ in } L^1 \\ |\nabla u_k|(\Omega) \longrightarrow |\nabla u|(\mathbb{R}^n) \end{cases}$

(Ω Lip)



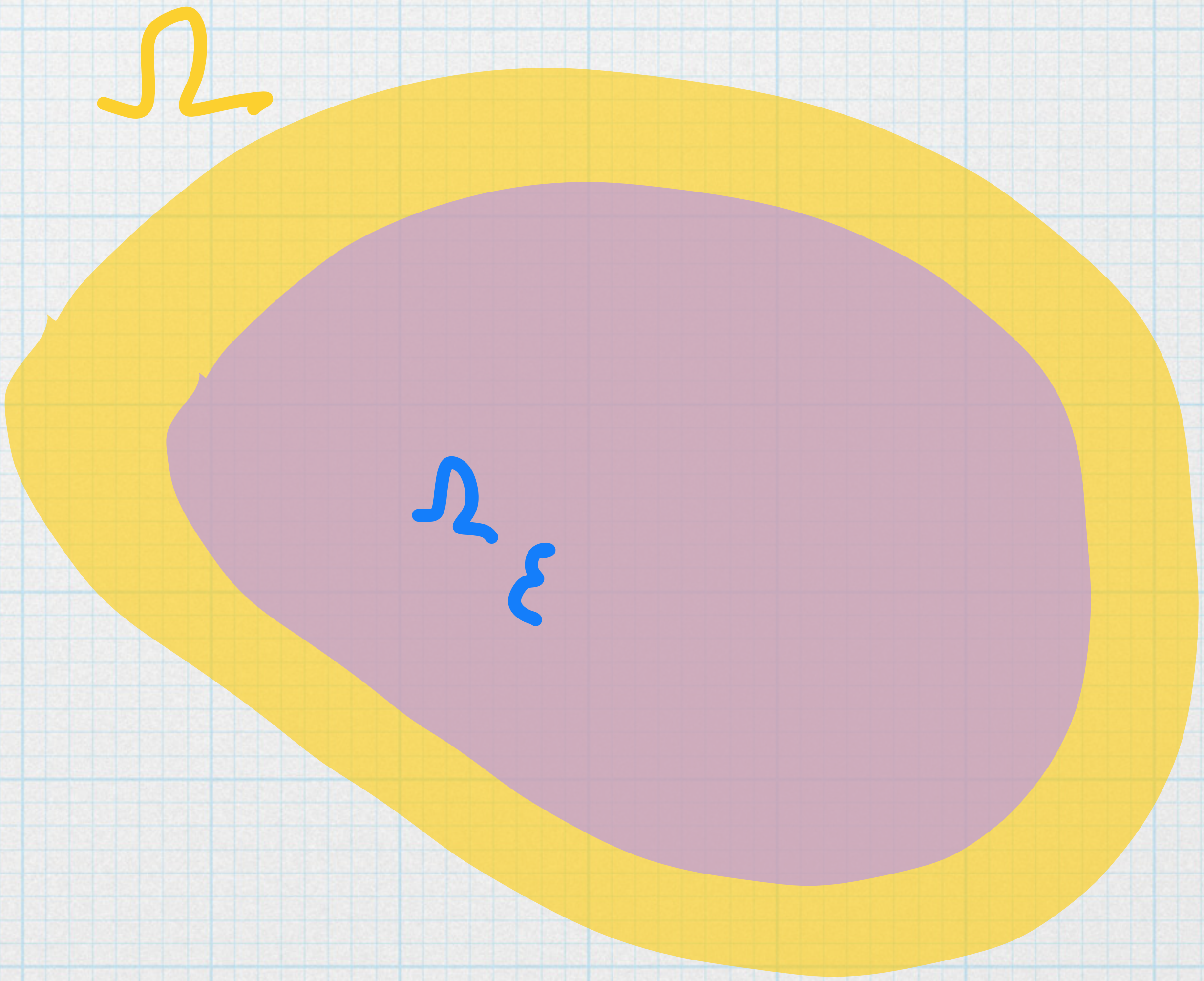
Remark 3 $\beta \in \mathbb{R}$

$$\Lambda(\Omega,\beta) = \inf_{E \subseteq \Omega} \frac{Per(E;\Omega) + \min\{\beta,1\}\,\mathcal{H}^{N-1}(\partial\Omega \cap \partial E)}{|E|} \quad \Big\} = J(\chi_E)$$

Remark 4

If $\beta < -1$, then $\Lambda(\Omega,\beta) = -\infty$, $\forall \Omega$ Lipschitz

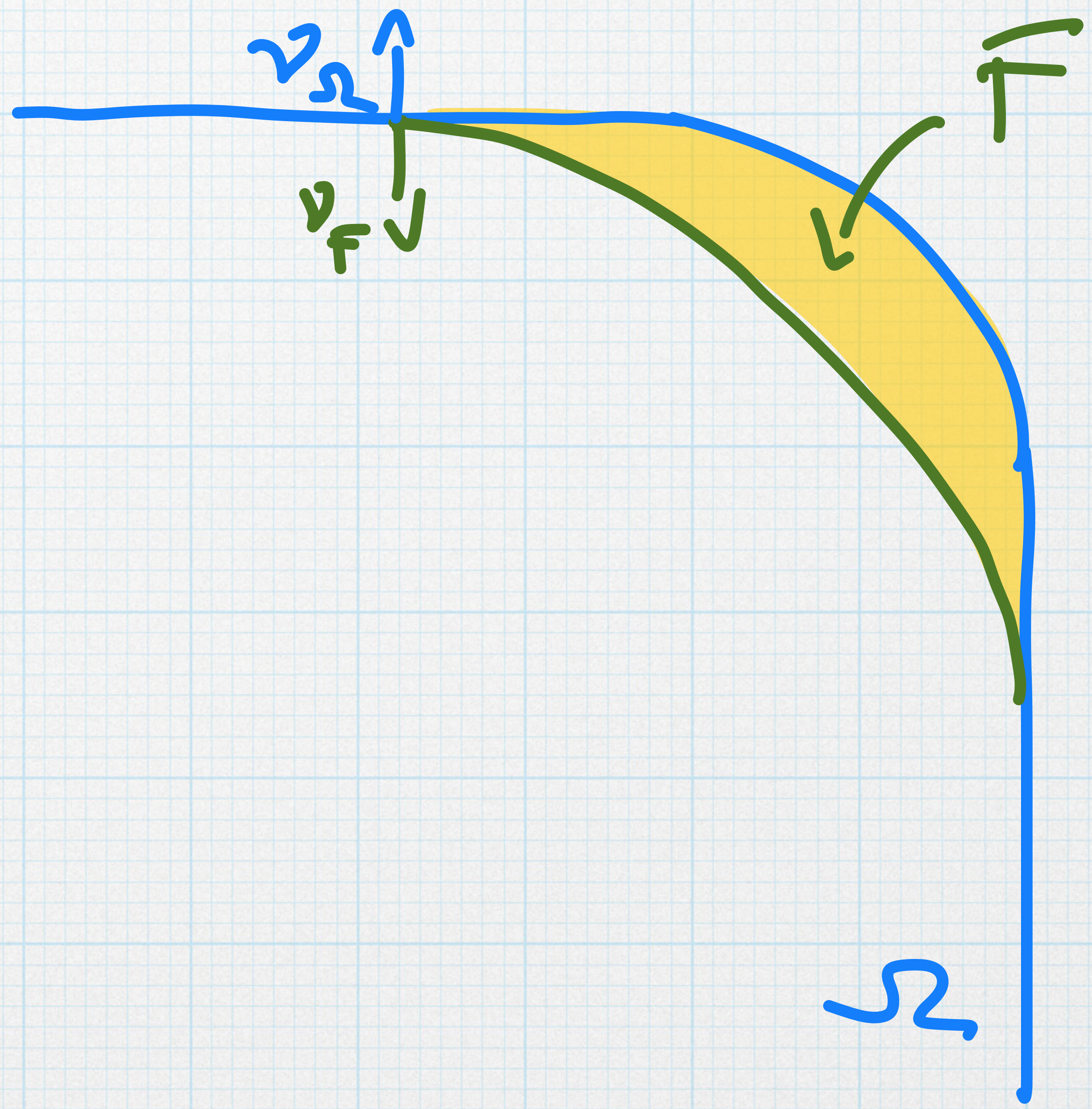
Indeed, $\exists\, \Omega_\varepsilon \subset \Omega$: $\begin{cases} |\Omega \setminus \Omega_\varepsilon| \to 0 \\ Per(\Omega_\varepsilon) \leq Per(\Omega) + \varepsilon \end{cases}$



$$\Lambda(\Omega,\beta) \leq J(\Omega \setminus \Omega_\varepsilon) \leq \frac{(1+\beta)P(\Omega) + \varepsilon}{|\Omega \setminus \Omega_\varepsilon|} \longrightarrow -\infty$$

Remark 5 $\quad \beta = -1$

$$J(\chi_E) = \frac{P_{er}(E;\Omega) - \mathcal{H}^{N-1}(\partial E \cap \partial \Omega)}{|E|}$$



$$\exists \min_{|E|=k} J(\chi_E) = J(\chi_F)$$

1) F has constant mean curvature in Ω

2) $\langle \nu_F, \nu_\Omega \rangle = -1$

If Ω has bounded mean curvature, then $\inf_{E \subseteq \Omega} J(\chi_E)$ is finite but, in general, not achieved

Remark 6: Hypotheses on $\partial\Omega$

The regularity on $\partial\Omega$ in order to get $\Lambda(\Omega,\beta)$ finite depends on β

$\beta \geq 0$: Ω Lipschitz

$-1 < \beta < 0$ $\partial\Omega$ C^1

Trace inequality:

$$\int_{\partial\Omega} |v| \leq (1+\varepsilon)|\nabla v|(\Omega) + C_2 \int_\Omega |v| \qquad \forall \varepsilon > 0,\ \forall v \in BV(\Omega)$$

Theorem 1 (D.P. - Nitsch - Oliva - Trombetti)

Let $\beta > -1$. $\exists$ a minimizer $v \in BV(\Omega)$ of $\Lambda(\Omega, \beta)$

Moreover,

$$\Lambda(\Omega, \beta) = J(\chi_{\{v > t\}}) = \min_{E \subseteq \Omega} \frac{Per(E; \Omega) + \beta \mathcal{H}^{n-1}(\partial E \cap \partial\Omega)}{|E|}$$

for some $t \in \mathbb{R}$.

Theorem 2 (D.P. - Nitsch - Oliva - Trombetti)

Let $\beta > -1$. Then, given

$$J_p(u) = \frac{\int_\Omega |\nabla u|^p + \beta \int_{\partial\Omega} |u|^p}{\int_\Omega |u|^p}, \quad J(u) = \frac{|\nabla u|(\Omega) + \min\{1,\beta\} \int_{\partial\Omega} |u|}{\int_\Omega |u|}$$

J_p Γ-converges to J in $BV(\Omega)$ as $p \to 1$

Corollary

$$\lim_{p \to 1} \lambda(\Omega, p, \beta) = \Lambda(\Omega, \beta)$$

Moreover, if $u_p \in W^{1,p}(\Omega)$ is a minimizer of J_p, then $u_p \to u$ weak* in $BV(\Omega)$, where u is a minimizer of J

Isoperimetric inequality for $\Lambda(\Omega,\beta)$

Theorem 3 (D.P.-Nitsch-Oliva-Trombetti)

Faber-Krahn inequality:

If $\beta \geq 0$, then $\Lambda(\Omega,\beta) \geq \Lambda(B,\beta)$, with B ball such that $|B|=|\Omega|$

Barket inequality:

If $\beta < 0$ then $\Lambda(\Omega,\beta) \leq \Lambda(B,\beta)$, with B ball s.t. $|B|=|\Omega|$

Proof

(1) $$\Lambda(B_R,\beta) = \underbrace{\min\{\beta,1\}}_{\hat\beta}\frac{N}{R} \Big) =: h(B_R)$$

Case $\beta \geq 0$

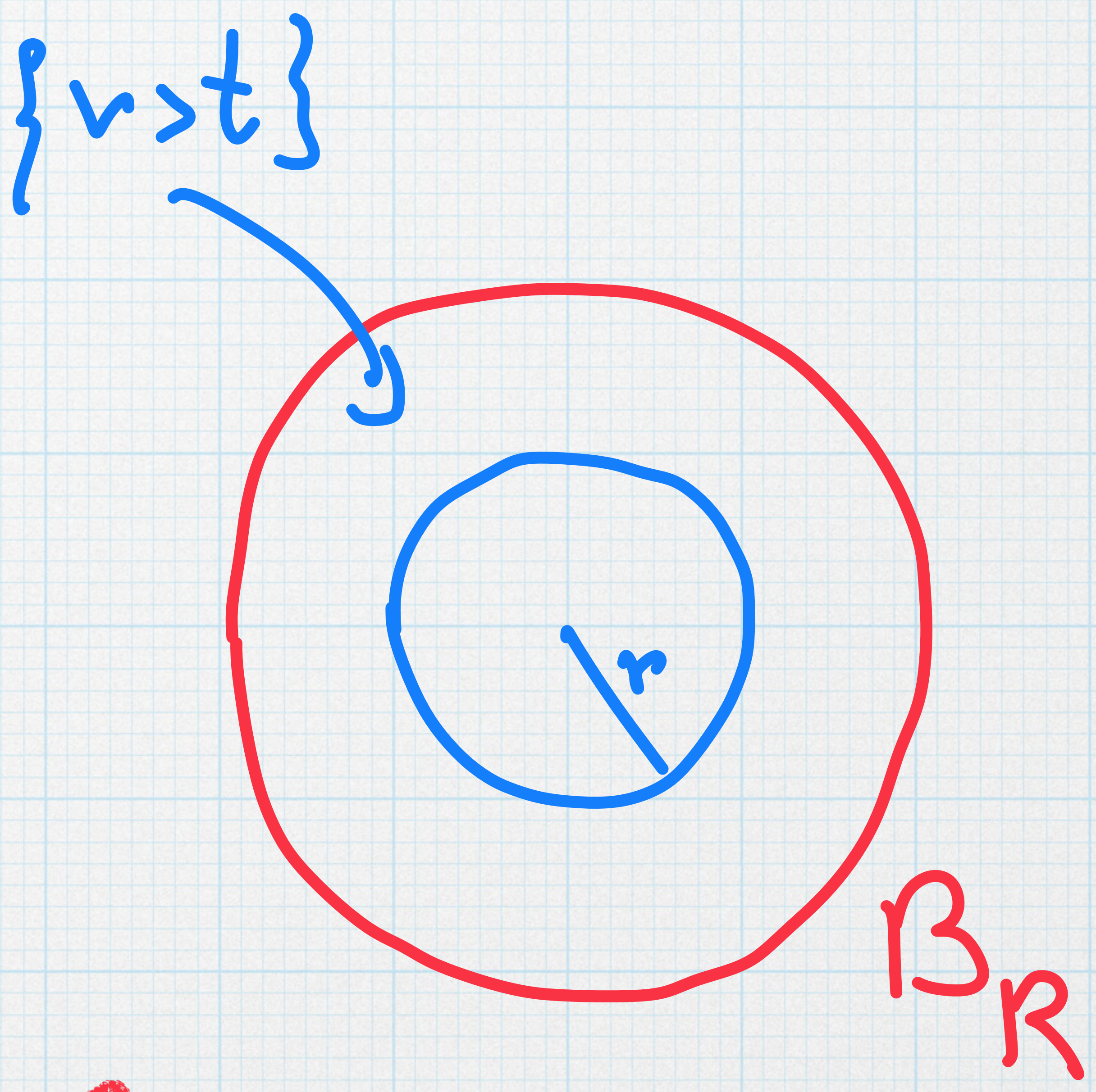
$$\frac{P_{er}(E;B_R) + \hat\beta \mathcal{H}^{N-1}(\partial B_R \cap \partial E)}{|E|} \geq \hat\beta \frac{P_{er}(E)}{|E|} \geq \hat\beta \frac{P_{er}(E^*)}{|E^*|} \geq \hat\beta \frac{N}{R}$$

$$\Rightarrow \quad \Lambda(B_R,\beta) \geq \hat\beta \frac{N}{R}$$

$\leq$ by choosing $E = B_R$

Case $-1<\beta<0$

Let v_p be a radial minimizer of J_p



- $v_p > 0$, $v_p(x) = v_p(r)$ radially increasing
- $v_p \to v$ minimizer for $\Lambda(B_R, \beta)$
- $J_p(v_p) \longrightarrow \Lambda(B_R, \beta) = J\left(\chi_{\{v>t\}}\right)$

N-dimensional spherical shells

$$\Lambda(B_R, \beta) = N \frac{r^{N-1} + \beta R^{N-1}}{R^N - r^N}, \text{ for some } r$$

$r_{min} = 0$

② $\beta \geq 0$

isop. inequality

$$J(x_\epsilon) \geq \hat{\beta}\,\frac{P_{\ell_2}(\epsilon)}{|\epsilon|} \geq \hat{\beta}\,\frac{P_{\ell_2}(B)}{|B|} = \Lambda(B,\beta) \implies \Lambda(\Omega,\beta) \geq \Lambda(B,\beta)$$

$-1 < \beta < 0$

isop. inequality

$$\Lambda(\Omega,\beta) \leq \beta\,\frac{P_{\ell_2}(\Omega)}{|\Omega|} \leq \beta\,\frac{P_{\ell_2}(B)}{|B|} = \Lambda(B,\beta)$$

Thank you!